

26. Probability Functions

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26. Probability Functions

Mathematical Properties³

26.1. Probability Functions: Definitions and Properties

Univariate Cumulative Distribution Functions

A real-valued function $F(x)$ is termed a (univariate) cumulative distribution function (c.d.f.) or simply distribution function if

- i) $F(x)$ is non-decreasing, i.e., $F(x_1) \leq F(x_2)$ for $x_1 \leq x_2$
- ii) $F(x)$ is everywhere continuous from the right, i.e., $F(x) = \lim_{\epsilon \rightarrow 0+} F(x + \epsilon)$
- iii) $F(-\infty) = 0, F(\infty) = 1$.

The function $F(x)$ signifies the probability of the event " $X \leq x$ " where X is a random variable, i.e., $Pr\{X \leq x\} = F(x)$, and thus describes the c.d.f. of X . The two principal types of distribution functions are termed *discrete* and *continuous*.

Discrete Distributions: Discrete distributions are characterized by the random variable X taking on an enumerable number of values . . . , x_{-1}, x_0, x_1, \dots with point probabilities

$$p_n = Pr\{X = x_n\} \geq 0$$

which need only be subject to the restriction

$$\sum_n p_n = 1.$$

The corresponding distribution function can then be written

$$26.1.1 \quad F(x) = Pr\{X \leq x\} = \sum_{x_n \leq x} p_n$$

³ Comment on notation and conventions.

a. We follow the customary convention of denoting a random variable by a capital letter, i.e., X , and using the corresponding lower case letter, i.e., x , for a particular value that the random variable assumes.

b. For statistical applications it is often convenient to have tabulated the "upper tail area," $1 - F(x)$, or the c.d.f. for $|X|$, $F(x) - F(-x)$, instead of simply the c.d.f. $F(x)$. We use the notation P to indicate the c.d.f. of X , $Q = 1 - P$ to indicate the "upper tail area" and $A = P - Q$ to denote the c.d.f. of $|X|$. In particular we use $P(x)$, $Q(x)$, and $A(x)$ to denote the corresponding functions for the normal or Gaussian probability function, see 26.2.2-26.2.4. When these distributions depend on other parameters, say θ_1 and θ_2 , we indicate this by writing $P(x|\theta_1, \theta_2)$, $Q(x|\theta_1, \theta_2)$, or $A(x|\theta_1, \theta_2)$. For example the chi-square distribution 26.4 depends on the parameter ν and the tabulated function is written $Q(\chi^2|\nu)$.

where the summation is over all values of x for which $x_n \leq x$. The set $\{x_n\}$ of values for which $p_n > 0$ is termed the domain of the random variable X . A discrete distribution of a random variable is called a *lattice distribution* if there exist numbers a and $b \neq 0$ such that every possible value of X can be represented in the form $a + bn$ where n takes on only integral values. A summary of some properties of certain discrete distributions is presented in 26.1.19-26.1.24.

Continuous Distributions. Continuous distributions are characterized by $F(x)$ being absolutely continuous. Hence $F(x)$ possesses a derivative $F'(x) = f(x)$ and the c.d.f. can be written

$$26.1.2 \quad F(x) = Pr\{X \leq x\} = \int_{-\infty}^x f(t) dt.$$

The derivative $f(x)$ is termed the *probability density function* (p.d.f.) or *frequency function*, and the values of x for which $f(x) > 0$ make up the domain of the random variable X . A summary of some properties of certain selected continuous distributions is presented in 26.1.25-26.1.34.

Multivariate Probability Functions

The real-valued function $F(x_1, x_2, \dots, x_n)$ defines an n -variate cumulative distribution function if

- i) $F(x_1, x_2, \dots, x_n)$ is a non-decreasing function for each x_i
- ii) $F(x_1, x_2, \dots, x_n)$ is continuous from the right in each x_i ; i.e., $F(x_1, x_2, \dots, x_n) = \lim_{\epsilon \rightarrow 0+} F(x_1, \dots, x_i + \epsilon, \dots, x_n)$
- iii) $F(x_1, x_2, \dots, x_n) = 0$ when any $x_i = -\infty$; $F(\infty, \infty, \dots, \infty) = 1$.
- iv) $F(x_1, x_2, \dots, x_n)$ assigns nonnegative probability to the event $x_1 < X_1 \leq x_1 + h_1, x_2 < X_2 \leq x_2 + h_2, \dots, x_n < X_n \leq x_n + h_n$ for all x_1, x_2, \dots, x_n and all nonnegative h_1, h_2, \dots, h_n , e.g., for $n=2$, $F(x_1 + h_1, x_2 + h_2) - F(x_1, x_2 + h_2) - F(x_1 + h_1, x_2) + F(x_1, x_2) \geq 0$ and in general for $x_i < X_i \leq x_i + h_i$ ($i=1, 2, \dots, n$), the k th order difference $\Delta_k F(x_1, x_2, \dots, x_n) > 0$ for $k=1, 2, \dots, n$.

*See page 11.

The joint probability of the event $X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n$ is $F(x_1, x_2, \dots, x_n)$. Analogous to the one-dimensional case, discrete distributions assign all probability to an enumerable set of

vectors (x_1, x_2, \dots, x_n) and continuous distributions are characterized by absolute continuity of $F(x_1, x_2, \dots, x_n)$.

Characteristics of distribution functions: Moments, characteristic functions, cumulants

		Continuous distributions	Discrete distributions
26.1.3	n^{th} moment about origin	$\mu'_n = \int_{-\infty}^{\infty} x^n f(x) dx$	$\mu'_n = \sum_i x_i^n p_i$
26.1.4	mean	$m = \mu'_1 = \int_{-\infty}^{\infty} x f(x) dx$	$m = \mu'_1 = \sum_i x_i p_i$
26.1.5	variance	$\sigma^2 = \mu'_2 - m^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x) dx$	$\sigma^2 = \mu'_2 - m^2 = \sum_i (x_i - m)^2 p_i$
26.1.6	n^{th} central moment	$\mu_n = \int_{-\infty}^{\infty} (x-m)^n f(x) dx$	$\mu_n = \sum_i (x_i - m)^n p_i$
26.1.7	expected value operator for the function $g(x)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$	$E[g(X)] = \sum_i g(x_i) p_i$
26.1.8	characteristic function of X	$\phi(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$	$\phi(t) = E(e^{itX}) = \sum_i e^{itx_i} p_i$
26.1.9	characteristic function of $g(X)$	$\phi_g(t) = E(e^{itg(X)}) = \int_{-\infty}^{\infty} e^{itg(x)} f(x) dx$	$\phi_g(t) = E(e^{itg(X)}) = \sum_i e^{itg(x_i)} p_i$
26.1.10	inversion formula	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt$	$p_i = \frac{b}{2\pi} \int_{-\pi/b}^{\pi/b} e^{-itx_i} \phi(t) dt$ (lattice distributions only)

Relation of the Characteristic Function to Moments About the Origin

$$26.1.11 \quad \phi^{(n)}(0) = \left[\frac{d^n}{dt^n} \phi(t) \right]_{t=0} = i^n \mu'_n$$

Cumulant Function

$$26.1.12 \quad \ln \phi(t) = \sum_{n=0}^{\infty} \kappa_n \frac{(it)^n}{n!}$$

κ_n is called the n^{th} cumulant.

$$26.1.13 \quad \kappa_1 = m, \kappa_2 = \sigma^2, \kappa_3 = \mu_3, \kappa_4 = \mu_4 - 3\mu_2^2$$

Relation of Central Moments to Moments About the Origin

$$26.1.14 \quad \mu_n = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \mu'_j m^{n-j}$$

Coefficients of Skewness and Excess

$$26.1.15 \quad \gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{\mu_3}{\sigma^3} \quad (\text{skewness})$$

$$26.1.16 \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2} = \frac{\mu_4}{\sigma^4} - 3 \quad (\text{excess})$$

Occasionally coefficients of skewness and excess (or kurtosis) are given by

$$26.1.17 \quad \beta_1 = \gamma_1^2 = \left(\frac{\mu_3}{\sigma^3} \right)^2 \quad (\text{skewness})$$

$$26.1.18 \quad \beta_2 = \gamma_2 + 3 = \frac{\mu_4}{\sigma^4} \quad (\text{excess or kurtosis})$$

Name	Domain	Point Probabilities	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2	Characteristic function	Cumulants
26.1.19 Single point or degenerate	$x=c$ (c a constant)	$p=1$	$-\infty < c < +\infty$	c	0	-----	-----	$e^{i\lambda c}$	$\kappa_1 = \lambda, \kappa_r = 0$ for $r > 1$
26.1.20 Binomial	$x_s = s$, for $s=0, 1, 2, \dots, n$	$\binom{n}{s} p^s (1-p)^{n-s}$	$0 < p < 1$ ($q=1-p$)	np	npq	$\frac{q-p}{\sqrt{npq}}$	$\frac{1-6pq}{npq}$	$(q+pe^{it})^n$	$\kappa_1 = np$ $\kappa_{r+1} = pq \frac{d\kappa_r}{dp}$ for $r \geq 1$
26.1.21 Hypergeometric	$x_s = s$, for $s=0, 1, \dots, \min(n, N_1)$	$\frac{\binom{N_1}{s} \binom{N_2}{n-s}}{\binom{N_1+N_2}{n}}$	N_1 and N_2 integers, and $n \leq N_1+N_2$ ($N=N_1+N_2$, $p=N_1/N$ and $q=1-p=N_2/N$)	np	$npq \frac{(N-n)}{(N-1)}$	$\frac{q-p}{\sqrt{npq}} \frac{(N-1)}{(N-n)} \frac{(N-2n)}{(N-2)}$	Complicated	$\frac{\binom{N_2}{n}}{\binom{N}{n}} F(-n, -N_1; N_2-n+1; e^{it})$	Complicated
26.1.22 Poisson	$x_s = s$, for $s=0, 1, 2, \dots, \infty$	$\frac{e^{-m} m^s}{s!}$	$0 < m < \infty$	m	m	$m^{-\frac{1}{2}}$	m^{-1}	$e^m (e^{it}-1)$	$\kappa_r = m$ for $r=1, 2, \dots$
26.1.23 Negative binomial	$x_s = s$, for $s=0, 1, 2, \dots, \infty$	$\binom{n+s-1}{s} p^n (1-p)^s$	$n \geq 0$ and $0 < p < 1$ ($p=1/Q$, and $1-p=P/Q$)	nP	nPQ	$\frac{Q+P}{\sqrt{nPQ}}$	$\frac{1+6PQ}{nPQ}$	$(Q-Pe^{it})^{-n}$	$\kappa_1 = nP$ $\kappa_{r+1} = PQ \frac{d\kappa_r}{dQ}$ for $r \geq 1$
26.1.24 Geometric	$x_s = s$, for $s=0, 1, 2, \dots, \infty$	$p(1-p)^s$	$0 < p < 1$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{2-p}{\sqrt{1-p}}$	$6 + \frac{p^2}{1-p}$	$p[1-(1-p)e^{it}]^{-1}$	$\kappa_1 = \frac{1-p}{p}$ $\kappa_{r+1} = -(1-p) \frac{d\kappa_r}{dp}$ $r \geq 1$

Some one-dimensional continuous distribution functions

	Name	Domain	Probability Density Function $f(x)$	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2	Characteristic function	Cumulants
26.1.25	Error function	$-\infty < x < \infty$	$\frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$	$0 < h < \infty$	0	$\frac{1}{2h^2}$	0	0	$e^{-\frac{t^2}{4h^2}}$	$\kappa_1 = 0, \kappa_2 = \frac{1}{2h^2}$ $\kappa_n = 0$ for $n > 2$
26.1.26	Normal	$-\infty < x < \infty$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$	$-\infty < m < \infty$ $0 < \sigma < \infty$	m	σ^2	0	0	$e^{imt - \frac{\sigma^2 t^2}{2}}$	$\kappa_1 = m, \kappa_2 = \sigma^2, \kappa_n = 0$ for $n > 2$
26.1.27	Cauchy	$-\infty < x < \infty$	$\frac{1}{\pi\beta} \frac{1}{1 + \left(\frac{x-\alpha}{\beta}\right)^2}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	not defined	not defined	not defined	not defined	$e^{i\alpha t - \beta t }$	not defined
26.1.28	Exponential	$\alpha \leq x < \infty$	$\frac{1}{\beta} e^{-\left(\frac{x-\alpha}{\beta}\right)}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	$\alpha + \beta$	β^2	2	6	$e^{i\alpha t} (1 - i\beta t)^{-1}$	$\kappa_1 = \alpha + \beta, \kappa_n = \beta^n \Gamma(n)$ for $n > 1$
26.1.29	Laplace, or double exponential	$-\infty < x < \infty$	$\frac{1}{2\beta} e^{-\left \frac{x-\alpha}{\beta}\right }$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	α	$2\beta^2$	0	3	$e^{i\alpha t} (1 + \beta^2 t^2)^{-1}$	$\kappa_1 = \alpha, \kappa_2 = 2\beta^2$ $\kappa_{2n+1} = 0, \kappa_{2n} = \frac{(2n)!}{n} \beta^{2n}$ for $n = 1, 2, \dots$
26.1.30	Extreme-Value, ⁴ (Fisher-Tippett Type I or doubly exponential)	$-\infty < x < \infty$	$\frac{1}{\beta} \exp(-y - e^{-y})$ with $y = \frac{x-\alpha}{\beta}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	$\alpha + \gamma\beta$	$\frac{(\pi\beta)^2}{6}$	1.3	2.4	$\Gamma(1 - i\beta t) e^{i\alpha t}$	$\kappa_1 = \gamma, \kappa_2 = \frac{(\pi\beta)^2}{6}$ $\kappa_n = \beta^n \Gamma(n) \sum_{r=1}^{\infty} \frac{1}{r^n}$ for $n > 2$
26.1.31	Pearson Type III	$\alpha \leq x < \infty$	$\frac{1}{\beta \Gamma(p)} y^{p-1} e^{-y}$ with $y = \frac{x-\alpha}{\beta}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$ $0 < p < \infty$	$\alpha + p\beta$	$p\beta^2$	$\frac{2}{\sqrt{p}}$	$6/p$	$e^{i\alpha t} (1 - i\beta t)^{-p}$	$\kappa_1 = \alpha + p\beta, \kappa_n = \beta^n p \Gamma(n)$ for $n > 1$
26.1.32	Gamma distribution	$0 \leq x < \infty$	$\frac{1}{\Gamma(p)} x^{p-1} e^{-x}$	$0 < p < \infty$	p	p	$\frac{2}{\sqrt{p}}$	$6/p$	$(1 - it)^{-p}$	$\kappa_1 = p, \kappa_n = p \Gamma(n)$ for $n > 1$
26.1.33	Beta distribution	$0 \leq x \leq 1$	$\frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}$	$1 \leq a < \infty$ $1 \leq b < \infty$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	$\frac{2(a-b)}{(a+b+2)}$	See footnote 5.	$M(a, a+b, it)$	
26.1.34	Rectangular, or uniform	$m - \frac{h}{2} \leq x \leq m + \frac{h}{2}$	$\frac{1}{h}$	$-\infty < m < \infty$ $0 < h < \infty$	m	$\frac{h^2}{12}$	0	-1.2	$\frac{2}{ht} \sin\left(\frac{ht}{2}\right) e^{imt}$	$\kappa_1 = m, \kappa_{2n+1} = 0$ $\kappa_{2n} = \frac{h^{2n} B_{2n}}{2n}$ B_{2n} (Bernoulli numbers), $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \dots$ *

⁴ γ (Euler's constant) = .57721 56649⁵ $\gamma_2 = -\sqrt{\frac{a+b+1}{ab}} \left\{ \frac{3(a+b+1)[2(a+b)^2 + ab(a+b-6)]}{ab(a+b+2)(a+b+3)} - 3 \right\}$.

* See page 11.

Inequalities for distribution functions

($F(x)$ denotes the c.d.f. of the random variable X and t denotes a positive constant; further m is always assumed to be finite and all expectations are assumed to exist.)

Inequality	Conditions
26.1.35 $Pr\{g(X) \geq t\} \leq E[g(X)]/t$	(i) $g(X) \geq 0$
26.1.36 $Pr\{X \geq t\} \leq m/t$ $F(t) \geq 1 - \frac{m}{t}$	(i) $Pr\{X < 0\} = 0$ (ii) $E(X) = m$
26.1.37 $Pr\{ X - m \geq t\sigma\} \leq 1/t^2$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{1}{t^2}$	(i) $E(X) = m$ (ii) $E(X - m)^2 = \sigma^2$ *
26.1.38 $Pr\{ \bar{X} - \bar{m} \geq t\bar{\sigma}\} \leq \frac{1}{nt^2}$	(i) $E(X_i) = m_i$ (ii) $E(X_i - m_i)^2 = \sigma_i^2$ (iii) $E[(X_i - m_i)(X_j - m_j)] = 0 (i \neq j)$ (iv) $\bar{X} = \sum_{i=1}^n \frac{X_i}{n}$ $\bar{m} = \sum_{i=1}^n \frac{m_i}{n}, \bar{\sigma} = \left[\sum_{i=1}^n \frac{\sigma_i^2}{n} \right]^{1/2}$
26.1.39 $Pr\{ X - m \geq t\sigma\} \leq \frac{4}{9} \left\{ \frac{1 + \left(\frac{m - x_0}{\sigma} \right)^2}{\left(t - \left \frac{m - x_0}{\sigma} \right \right)^2} \right\}$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{4}{9} \left\{ \frac{1 + \left(\frac{m - x_0}{\sigma} \right)^2}{\left(t - \left \frac{m - x_0}{\sigma} \right \right)^2} \right\}$	(i) $E(X - m)^2 = \sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0^6
26.1.40 $Pr\{ X - m \geq t\sigma\} \leq 4/9t^2$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{4}{9t^2}$	(i) $E(X - m)^2 = \sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0^6 (iv) $m = x_0$
26.1.41 $Pr\{ X - m \geq t\sigma\} \leq \frac{\mu_4 - \sigma^4}{\mu_4 + t^4\sigma^4 - 2t^2\sigma^4}$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{\mu_4 - \sigma^4}{\mu_4 + t^4\sigma^4 - 2t^2\sigma^4}$	(i) $E(X - m)^2 = \sigma^2$ (ii) $E(X - m)^4 = \mu_4$

⁶ x_0 is such that $F'(x_0) > F'(x)$ for $x \neq x_0$.

26.2. Normal or Gaussian Probability Function

$$26.2.1 \quad Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$26.2.2 \quad P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt = \int_{-\infty}^x Z(t) dt$$

$$26.2.3 \quad Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt = \int_x^{\infty} Z(t) dt$$

$$26.2.4 \quad A(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-t^2/2} dt = \int_{-x}^x Z(t) dt$$

$$26.2.5 \quad P(x) + Q(x) = 1$$

$$26.2.6 \quad P(-x) = Q(x)$$

$$26.2.7 \quad A(x) = 2P(x) - 1$$

Probability Integral with Mean m and Variance σ^2

A random variable X is said to be normally distributed with mean m and variance σ^2 if the probability that X is less than or equal to x is given by

26.2.8

$$Pr\{X \leq x\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-m)^2}{2\sigma^2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt = P\left(\frac{x-m}{\sigma}\right).$$

The corresponding probability density function is

26.2.9

$$\frac{\partial}{\partial x} P\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sigma} Z\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

and is symmetric around m , i.e.

$$Z\left(\frac{m+x}{\sigma}\right) = Z\left(\frac{m-x}{\sigma}\right).$$

The inflexion points of the probability density function are at $m \pm \sigma$.

Power Series ($x \geq 0$)

$$26.2.10 \quad P(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! 2^n (2n+1)}$$

$$26.2.11 \quad P(x) = \frac{1}{2} + Z(x) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

Asymptotic Expansions ($x > 0$)

$$26.2.12 \quad Q(x) = \frac{Z(x)}{x} \left\{ 1 - \frac{1}{x^2} + \frac{1 \cdot 3}{x^4} + \cdots + \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{x^{2n}} \right\} + R_n$$

where

$$R_n = (-1)^{n+1} 1 \cdot 3 \cdots (2n+1) \int_x^{\infty} \frac{Z(t)}{t^{2n+2}} dt$$

which is less in absolute value than the first neglected term.

26.2.13

$$Q(x) \sim \frac{Z(x)}{x} \left\{ 1 - \frac{a_1}{x^2+2} + \frac{a_2}{(x^2+2)(x^2+4)} - \frac{a_3}{(x^2+2)(x^2+4)(x^2+6)} + \cdots \right\}$$

where $a_1=1$, $a_2=1$, $a_3=5$, $a_4=9$, $a_5=129$ and the general term is

$$a_n = c_0 1 \cdot 3 \cdots (2n-1) + 2c_1 1 \cdot 3 \cdots (2n-3) + 2^2 c_2 1 \cdot 3 \cdots (2n-5) + \cdots + 2^{n-1} c_{n-1}$$

and c_s is the coefficient of t^{n-s} in the expansion of $t(t-1) \cdots (t-n+1)$.

Continued Fraction Expansions

26.2.14

$$Q(x) = Z(x) \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \frac{4}{x+} \cdots \right\} \quad (x > 0)$$

26.2.15

$$Q(x) = \frac{1}{2} - Z(x) \left\{ \frac{x}{1-} \frac{x^2}{3+} \frac{2x^2}{5-} \frac{3x^2}{7+} \frac{4x^2}{9-} \cdots \right\} \quad (x \geq 0)$$

Polynomial and Rational Approximations⁷ for $P(x)$ and $Z(x)$

$$0 \leq x < \infty$$

26.2.16

$$P(x) = 1 - Z(x)(a_1 t + a_2 t^2 + a_3 t^3) + \epsilon(x), \quad t = \frac{1}{1+px}$$

$$|\epsilon(x)| < 1 \times 10^{-5}$$

$$p = .33267 \quad a_1 = .43618 \quad 36 \\ a_2 = -.12016 \quad 76 \\ a_3 = .93729 \quad 80$$

26.2.17

$$P(x) = 1 - Z(x)(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) + \epsilon(x), \quad t = \frac{1}{1+px}$$

$$|\epsilon(x)| < 7.5 \times 10^{-8}$$

$$p = .23164 \quad 19 \\ b_1 = .31938 \quad 1530 \quad b_4 = -1.82125 \quad 5978 \\ b_2 = -.35656 \quad 3782 \quad b_5 = 1.33027 \quad 4429 \\ b_3 = 1.78147 \quad 7937$$

26.2.18

$$P(x) = 1 - \frac{1}{2} (1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4)^{-4} + \epsilon(x)$$

$$|\epsilon(x)| < 2.5 \times 10^{-4}$$

$$c_1 = .196854 \quad c_3 = .000344 \\ c_2 = .115194 \quad c_4 = .019527$$

26.2.19

$$P(x) = 1 - \frac{1}{2} (1 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 x^5 + d_6 x^6)^{-16} + \epsilon(x)$$

$$|\epsilon(x)| < 1.5 \times 10^{-7}$$

$$d_1 = .04986 \quad 73470 \quad d_4 = .00003 \quad 80036 \\ d_2 = .02114 \quad 10061 \quad d_5 = .00004 \quad 88906 \\ d_3 = .00327 \quad 76263 \quad d_6 = .00000 \quad 53830$$

$$26.2.20 \quad Z(x) = (a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6)^{-1} + \epsilon(x)$$

$$|\epsilon(x)| < 2.7 \times 10^{-3}$$

$$a_0 = 2.490895 \quad a_4 = -.024393 \\ a_2 = 1.466003 \quad a_6 = .178257$$

⁷ Based on approximations in C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

26.2.21

$$Z(x) = (b_0 + b_2x^2 + b_4x^4 + b_6x^6 + b_8x^8 + b_{10}x^{10})^{-1} + \epsilon(x)$$

$$|\epsilon(x)| < 2.3 \times 10^{-4}$$

$$b_0 = 2.50523 \quad 67 \quad b_6 = .13064 \quad 69$$

$$b_2 = 1.28312 \quad 04 \quad b_8 = -.02024 \quad 90$$

$$b_4 = .22647 \quad 18 \quad b_{10} = .00391 \quad 32$$

Rational Approximations ⁷ for x_p where $Q(x_p) = p$

$$0 < p \leq .5$$

26.2.22

$$x_p = t - \frac{a_0 + a_1 t}{1 + b_1 t + b_2 t^2} + \epsilon(p), \quad t = \sqrt{\ln \frac{1}{p^2}}$$

$$|\epsilon(p)| < 3 \times 10^{-3}$$

$$a_0 = 2.30753 \quad b_1 = .99229$$

$$a_1 = .27061 \quad b_2 = .04481$$

26.2.23

$$x_p = t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} + \epsilon(p), \quad t = \sqrt{\ln \frac{1}{p^2}}$$

$$|\epsilon(p)| < 4.5 \times 10^{-4}$$

$$c_0 = 2.515517 \quad d_1 = 1.432788$$

$$c_1 = .802853 \quad d_2 = .189269$$

$$c_2 = .010328 \quad d_3 = .001308$$

Bounds Useful as Approximations to the Normal Distribution Function

26.2.24

$$P(x) \leq \begin{cases} P_1(x) = \frac{1}{2} + \frac{1}{2} (1 - e^{-2x^2/\pi})^{\frac{1}{2}} & (x > 0) \\ P_2(x) = 1 - \frac{(4 + x^2)^{\frac{1}{2}} - x}{2} (2\pi)^{-\frac{1}{2}} e^{-x^2/2} & (x > 1.4) \end{cases}$$

26.2.25

$$P(x) \geq \begin{cases} P_3(x) = \frac{1}{2} + \frac{1}{2} \left(1 - e^{-2x^2/\pi} - \frac{2(\pi-3)}{3\pi^2} x^4 e^{-x^2/2} \right)^{\frac{1}{2}} & (x > 0) \\ P_4(x) = 1 - \frac{1}{x} (2\pi)^{-\frac{1}{2}} e^{-x^2/2} & (x > 2.2) \end{cases}$$

See **Figure 26.1** for error curves.

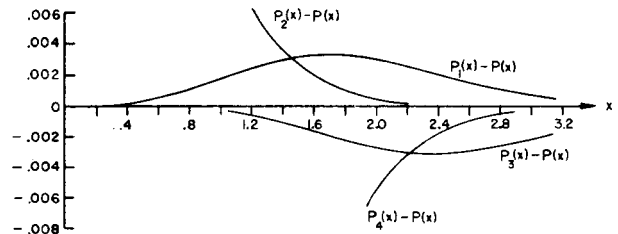


FIGURE 26.1. Error curves for bounds on normal distribution.

Derivatives of the Normal Probability Density Function

$$26.2.26 \quad Z^{(m)}(x) = \frac{d^m}{dx^m} Z(x)$$

Differential Equation

$$26.2.27 \quad Z^{(m+2)}(x) + xZ^{(m+1)}(x) + (m+1)Z^{(m)}(x) = 0$$

Value at $x=0$

26.2.28

$$Z^{(m)}(0) = \begin{cases} \frac{(-1)^{m/2} m!}{\sqrt{2\pi} 2^{m/2} \left(\frac{m}{2}\right)!} & \text{for } m=2r, r=0, 1, \dots \\ 0 & \text{for odd } m > 0 \end{cases}$$

Relation of $P(x)$ and $Z^{(n)}(x)$ to Other Functions

Function	Relation	
26.2.29 Error function	$\operatorname{erf} x = 2P(x\sqrt{2}) - 1$	$(x \geq 0)$
26.2.30 Incomplete gamma function (special case)	$\frac{\gamma(\frac{1}{2}, x)}{\Gamma(\frac{1}{2})} = [2P(\sqrt{2x}) - 1]$	$(x \geq 0)$
26.2.31 Hermite polynomial	$He_n(x) = (-1)^n \frac{Z^{(n)}(x)}{Z(x)}$	
26.2.32 “	$H_n(x) = (-1)^n 2^{n/2} \frac{Z^{(n)}(x\sqrt{2})}{Z(x\sqrt{2})}$	
26.2.33 Hh function	$Hh_{-n}(x) = (-1)^{n-1} \sqrt{2\pi} Z^{(n-1)}(x)$	$(n > 0)$
26.2.34 “	$Hh_n(x) = \frac{(-1)^n}{n!} Hh_{-1}(x) \frac{d^n}{dx^n} \left(\frac{Q(x)}{Z(x)} \right)$	$(n > 0)$ *
26.2.35 Tetrachoric function	$\tau_n(x) = \frac{(-1)^{n-1}}{\sqrt{n!}} Z^{(n-1)}(x)$	
26.2.36 Confluent hypergeometric function (special case)	$M\left(\frac{1}{2}, \frac{3}{2}, -\frac{x^2}{2}\right) = \frac{\sqrt{2\pi}}{x} \left\{ P(x) - \frac{1}{2} \right\}$	$(x > 0)$
26.2.37 “	$M\left(1, \frac{3}{2}, \frac{x^2}{2}\right) = \frac{1}{xZ(x)} \left\{ P(x) - \frac{1}{2} \right\}$	$(x > 0)$
26.2.38 “	$M\left(\frac{2m+1}{2}, \frac{1}{2}, -\frac{x^2}{2}\right) = \frac{Z^{(2m)}(x)}{Z^{(2m)}(0)}$	$(x \geq 0)$
26.2.39 “	$M\left(\frac{2m+2}{2}, \frac{3}{2}, -\frac{x^2}{2}\right) = \frac{Z^{(2m-1)}(x)}{xZ^{(2m)}(0)}$	$(x \geq 0)$
26.2.40 Parabolic cylinder function	$U\left(-n - \frac{1}{2}, x\right) = e^{-\frac{1}{2}x^2} (-1)^n \frac{Z^{(n)}(x)}{Z(x)}$	$(n > 0)$

Repeated Integrals of the Normal Probability Integral

$$26.2.41 \quad I_n(x) = \int_x^\infty I_{n-1}(t) dt \quad (n \geq 0)$$

where $I_{-1}(x) = Z(x)$

26.2.42

$$I_{-n}(x) = \left(-\frac{d}{dx}\right)^{n-1} Z(x) = (-1)^{n-1} Z^{(n-1)}(x) \quad (n \geq -1)$$

$$26.2.43 \quad \left(\frac{d^2}{dx^2} + x \frac{dx}{dn} - n\right) I_n(x) = 0$$

26.2.44

$$(n+1)I_{n+1}(x) + xI_n(x) - I_{n-1}(x) = 0 \quad (n > -1)$$

*See page II.

26.2.45

$$I_n(x) = \int_x^\infty \frac{(t-x)^n}{n!} Z(t) dt = e^{-x^2/2} \int_0^\infty \frac{t^n}{n!} Z(t) dt \quad (n > -1)$$

$$26.2.46 \quad I_n(0) = I_{-n}(0) = \frac{1}{\left(\frac{n}{2}\right)! 2^{\frac{n+2}{2}}} \quad (n \text{ even})$$

Asymptotic Expansions of an Arbitrary Probability Density Function and Distribution Function

Let Y_i ($i=1, 2, \dots, n$) be n

independent random variables with mean m_i , variance σ_i^2 , and higher cumulants $\kappa_{r,i}$. Then asymptotic expansions with respect to n for the probability density and cumulative distribution function of

$$X = \frac{\sum_{i=1}^n (Y_i - m_i)}{\left(\sum_{i=1}^n \sigma_i^2\right)^{1/2}} \text{ are}$$

26.2.47

$$\begin{aligned} f(x) \sim Z(x) &- \left[\frac{\gamma_1}{6} Z^{(3)}(x) \right] + \left[\frac{\gamma_2}{24} Z^{(4)}(x) + \frac{\gamma_1^2}{72} Z^{(6)}(x) \right] \\ &- \left[\frac{\gamma_3}{120} Z^{(5)}(x) + \frac{\gamma_1 \gamma_2}{144} Z^{(7)}(x) + \frac{\gamma_1^3}{1296} Z^{(9)}(x) \right] \\ &+ \left[\frac{\gamma_4}{720} Z^{(6)}(x) + \frac{\gamma_2^2}{1152} Z^{(8)}(x) + \frac{\gamma_1 \gamma_3}{720} Z^{(8)}(x) \right. \\ &\quad \left. + \frac{\gamma_1^2 \gamma_2}{1728} Z^{(10)}(x) + \frac{\gamma_1^4}{31104} Z^{(12)}(x) \right] + \dots \end{aligned}$$

26.2.48

$$\begin{aligned} F(x) \sim P(x) &- \left[\frac{\gamma_1}{6} Z^{(2)}(x) \right] + \left[\frac{\gamma_2}{24} Z^{(3)}(x) + \frac{\gamma_1^2}{72} Z^{(5)}(x) \right] \\ &- \left[\frac{\gamma_3}{120} Z^{(4)}(x) + \frac{\gamma_1 \gamma_2}{144} Z^{(6)}(x) + \frac{\gamma_1^3}{1296} Z^{(8)}(x) \right] \\ &+ \left[\frac{\gamma_4}{720} Z^{(5)}(x) + \frac{\gamma_2^2}{1152} Z^{(7)}(x) + \frac{\gamma_1 \gamma_3}{720} Z^{(7)}(x) \right. \\ &\quad \left. + \frac{\gamma_1^2 \gamma_2}{1728} Z^{(9)}(x) + \frac{\gamma_1^4}{31104} Z^{(11)}(x) \right] + \dots \end{aligned}$$

where

$$\gamma_{r-2} = \frac{1}{n^{\frac{r}{2}-1}} \frac{\left(\frac{1}{n} \sum_{i=1}^n \kappa_{r,i}\right)}{\left(\frac{1}{n} \sum_{i=1}^n \sigma_i^2\right)^{r/2}}$$

Terms in brackets are terms of the same order with respect to n . When the Y_i have the same distribution, then $m_i = m$, $\sigma_i^2 = \sigma^2$, $\kappa_{r,i} = \kappa_r$ and

$$\gamma_{r-2} = \frac{1}{n^{\frac{r}{2}-1}} \left(\frac{\kappa_r}{\sigma^r}\right)$$

Asymptotic Expansion for the Inverse Function of an Arbitrary Distribution Function

Let the cumulative distribution function of $Y = \sum_{i=1}^n Y_i$ be denoted by $F(y)$. Then the (Cornish-Fisher) asymptotic expansion with respect to n for the value of y_p such that $F(y_p) = 1-p$ is

$$26.2.49 \quad y_p \sim m + \sigma w$$

where

$$\begin{aligned} w = x &+ [\gamma_1 h_1(x)] \\ &+ [\gamma_2 h_2(x) + \gamma_1^2 h_{11}(x)] \\ &+ [\gamma_3 h_3(x) + \gamma_1 \gamma_2 h_{12}(x) + \gamma_1^3 h_{111}(x)] \\ &+ [\gamma_4 h_4(x) + \gamma_2^2 h_{22}(x) + \gamma_1 \gamma_3 h_{13}(x) + \gamma_1^2 \gamma_2 h_{112}(x) \\ &\quad + \gamma_1^4 h_{1111}(x)] + \dots \end{aligned}$$

and

$$Q(x) = p, \quad \gamma_{r-2} = \frac{\kappa_r}{\kappa_2^{r/2}}, \quad r=3, 4, \dots$$

26.2.50

$$h_1(x) = \frac{1}{6} H e_2(x)$$

$$h_2(x) = \frac{1}{24} H e_3(x)$$

$$h_{11}(x) = -\frac{1}{36} [2H e_3(x) + H e_1(x)]$$

$$h_3(x) = \frac{1}{120} [H e_4(x)]$$

$$h_{12}(x) = -\frac{1}{24} [H e_4(x) + H e_2(x)]$$

$$h_{111}(x) = \frac{1}{324} [12H e_4(x) + 19H e_2(x)]$$

$$h_4(x) = \frac{1}{720} H e_5(x)$$

$$h_{22}(x) = -\frac{1}{384} [3H e_5(x) + 6H e_3(x) + 2H e_1(x)]$$

$$h_{13}(x) = -\frac{1}{180} [2H e_5(x) + 3H e_3(x)]$$

$$h_{112}(x) = \frac{1}{288} [14H e_5(x) + 37H e_3(x) + 8H e_1(x)]$$

$$h_{1111}(x) = -\frac{1}{7776} [252H e_5(x) + 832H e_3(x) + 227H e_1(x)]$$

Terms in brackets in 26.2.49 are terms of the same order with respect to n . The $H e_n(x)$ are the Hermite polynomials. (See chapter 22.)

$$26.2.51 \quad He_n(x) = (-1)^n \frac{Z^{(n)}(x)}{Z(x)} = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{2^m m! (n-2m)!} x^{n-2m}$$

In the following auxiliary table, the polynomial functions $h_1(x)$, $h_2(x)$. . . $h_{1111}(x)$ are tabulated for $p = .25, .1, .05, .025, .01, .005, .0025, .001, .0005$.

Auxiliary coefficients^a for use with Cornish-Fisher asymptotic expansion. 26.2.49

	p								
	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
x	.67449	1.28155	1.64485	1.95996	2.32635	2.57583	2.80703	3.09022	3.29053
$h_1(x)$	-.09084	.10706	.28426	.47358	.73532	.93915	1.14657	1.42491	1.63793
$h_2(x)$	-.07153	-.07249	-.02018	.06872	.23379	.39012	.57070	.84331	1.07320
$h_3(x)$.07663	.06106	-.01878	-.14607	-.37634	-.59171	-.83890	-1.21025	-1.52234
$h_4(x)$.00398	-.03464	-.04928	-.04410	-.00152	.06010	.14841	.30746	.46059
$h_5(x)$.00282	.14844	.17532	.10210	-.17621	-.53531	-1.02868	-1.89355	-2.71243
$h_6(x)$	-.01428	-.11629	-.11900	-.02937	.25195	.59757	1.06301	1.86787	2.62337
$h_7(x)$.00998	.00227	-.01082	-.02357	-.03176	-.02621	-.00666	.04591	.10950
$h_8(x)$	-.03285	.00776	.05985	.09659	.07888	-.01226	-.19116	-.59060	-1.03555
$h_9(x)$	-.05126	.01086	.09462	.16106	.16058	.05366	-.17498	-.70464	-1.30531
$h_{10}(x)$.14764	-.10856	-.39517	-.55856	-.32621	.35696	1.60445	4.29304	7.23307
$h_{1111}(x)$	-.06898	.09585	.25623	.31624	.07286	-.46534	-1.39199	-3.32708	-5.40702

^a From R. A. Fisher, Contributions to mathematical statistics, Paper 30 (with E. A. Cornish) Extrait de la Revue de l'Institut International de Statistique 4, 1-14 (1937) (with permission).

26.3. Bivariate Normal Probability Function

26.3.1

$$g(x, y, \rho) = [2\pi \sqrt{1-\rho^2}]^{-1} \exp -\frac{1}{2} \left(\frac{x^2 - 2\rho xy + y^2}{1-\rho^2} \right)$$

$$26.3.2 \quad g(x, y, \rho) = (1-\rho^2)^{-1/2} Z(x) Z\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)$$

26.3.3

$$L(h, k, \rho) = \int_h^\infty dx \int_k^\infty g(x, y, \rho) dy \\ = \int_h^\infty Z(x) dx \int_w^\infty Z(w) dw, \quad w = \left(\frac{k - \rho x}{\sqrt{1-\rho^2}} \right)$$

$$26.3.4 \quad L(-h, -k, \rho) = \int_{-\infty}^h dx \int_{-\infty}^k g(x, y, \rho) dy$$

$$26.3.5 \quad L(-h, k, -\rho) = \int_{-\infty}^h dx \int_k^\infty g(x, y, \rho) dy$$

$$26.3.6 \quad L(h, -k, -\rho) = \int_h^\infty dx \int_{-\infty}^k g(x, y, \rho) dy$$

$$26.3.7 \quad L(h, k, \rho) = L(k, h, \rho)$$

$$26.3.8 \quad L(-h, k, \rho) + L(h, k, -\rho) = Q(k)$$

$$26.3.9 \quad L(-h, -k, \rho) - L(h, k, \rho) = P(k) - Q(h)$$

26.3.10

$$* \quad 2[L(h, k, \rho) + L(h, k, -\rho) + P(h) - Q(k)] - 1 \\ = \int_{-h}^h dx \int_{-k}^k g(x, y, \rho) dy$$

Probability Function With Means m_x, m_y , Variances σ_x^2, σ_y^2 , and Correlation ρ

The random variables X, Y are said to be distributed as a bivariate Normal distribution with

means and variances (m_x, m_y) and (σ_x^2, σ_y^2) and correlation ρ if the joint probability that X is less than or equal to h and Y less than or equal to k is given by

26.3.11

$$Pr\{X \leq h, Y \leq k\} = \frac{1}{\sigma_x \sigma_y} \int_{-\infty}^{\frac{h-m_x}{\sigma_x}} \int_{-\infty}^{\frac{k-m_y}{\sigma_y}} g(s, t, \rho) ds dt \\ = L\left(-\left(\frac{h-m_x}{\sigma_x}\right), -\left(\frac{k-m_y}{\sigma_y}\right), \rho\right)$$

The probability density function is

26.3.12

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \frac{-Q}{2(1-\rho^2)} = \frac{1}{\sigma_x\sigma_y} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right)$$

where

$$Q = \frac{(x-m_x)^2}{\sigma_x^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}$$

Circular Normal Probability Density Function

26.3.13

$$\frac{1}{\sigma^2} g\left(\frac{x-m_x}{\sigma}, \frac{y-m_y}{\sigma}, 0\right) =$$

$$\frac{1}{2\pi\sigma^2} \exp -\frac{(x-m_x)^2 + (y-m_y)^2}{2\sigma^2}$$

Special Values of $L(h, k, \rho)$

$$26.3.14 \quad L(h, k, 0) = Q(h)Q(k)$$

$$26.3.15 \quad L(h, k, -1) = 0 \quad (h+k \geq 0)$$

$$26.3.16 \quad L(h, k, -1) = P(h) - Q(k) \quad (h+k \leq 0)$$

$$26.3.17 \quad L(h, k, 1) = Q(h) \quad (k \leq h)$$

$$26.3.18 \quad L(h, k, 1) = Q(k) \quad (k \geq h)$$

$$26.3.19 \quad L(0, 0, \rho) = \frac{1}{4} + \frac{\arcsin \rho}{2\pi}$$

 $L(h, k, \rho)$ as a Function of $L(h, 0, \rho)$

26.3.20

$$L(h, k, \rho) = L\left(h, 0, \frac{(\rho h - k)(\operatorname{sgn} h)}{\sqrt{h^2 - 2\rho h k + k^2}}\right) + L\left(k, 0, \frac{(\rho k - h)(\operatorname{sgn} k)}{\sqrt{h^2 - 2\rho h k + k^2}}\right) - \begin{cases} 0 & \text{if } h k > 0 \text{ or } h k = 0 \\ & \text{and } h + k \geq 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

where $\operatorname{sgn} h = 1$ if $h \geq 0$ and $\operatorname{sgn} h = -1$ if $h < 0$.

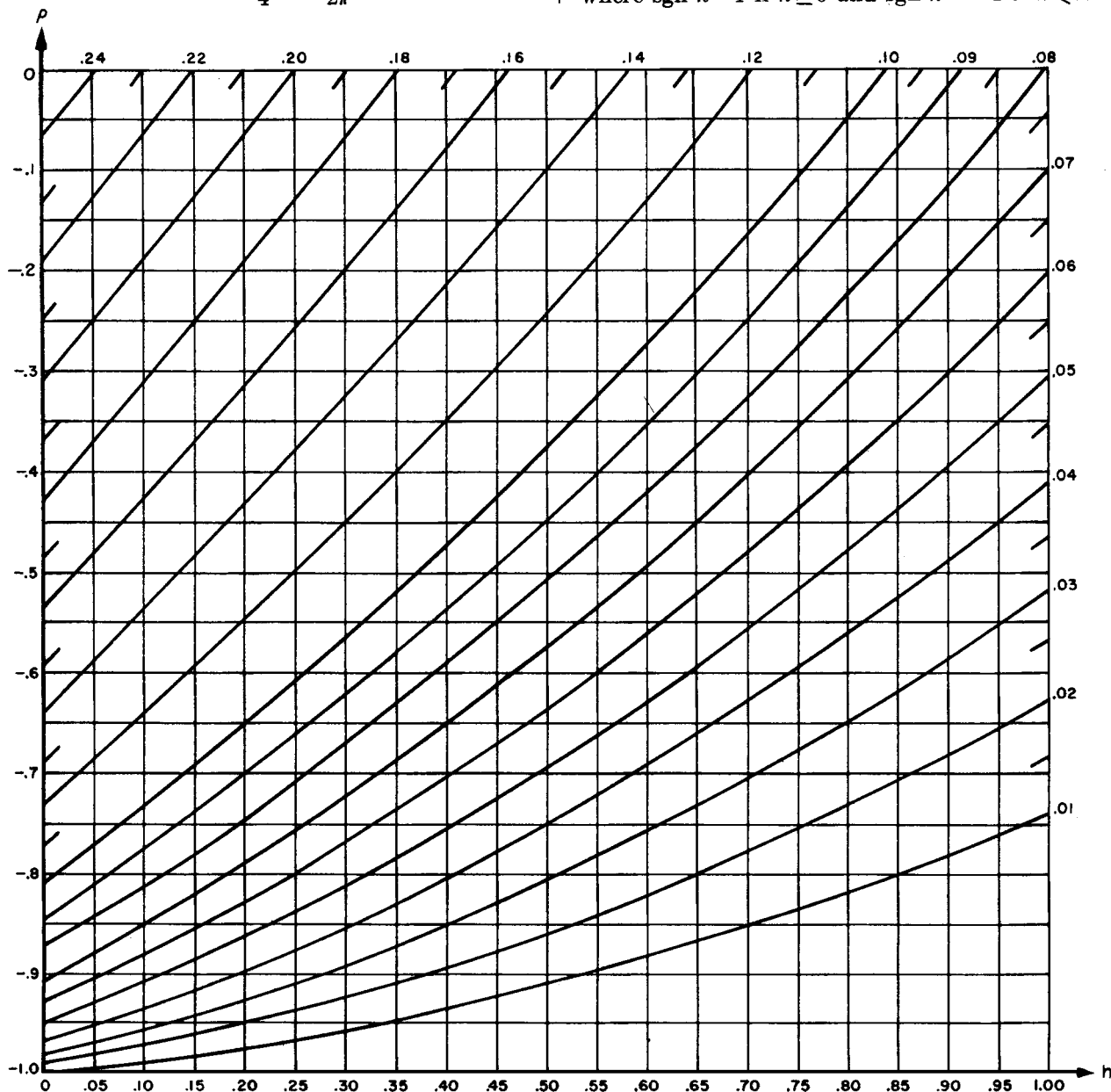


FIGURE 26.2. $L(h, 0, \rho)$ for $0 \leq h \leq 1$ and $-1 \leq \rho \leq 0$.

Values for $h < 0$ can be obtained using $L(h, 0, -\rho) = \frac{1}{2} - L(-h, 0, \rho)$.

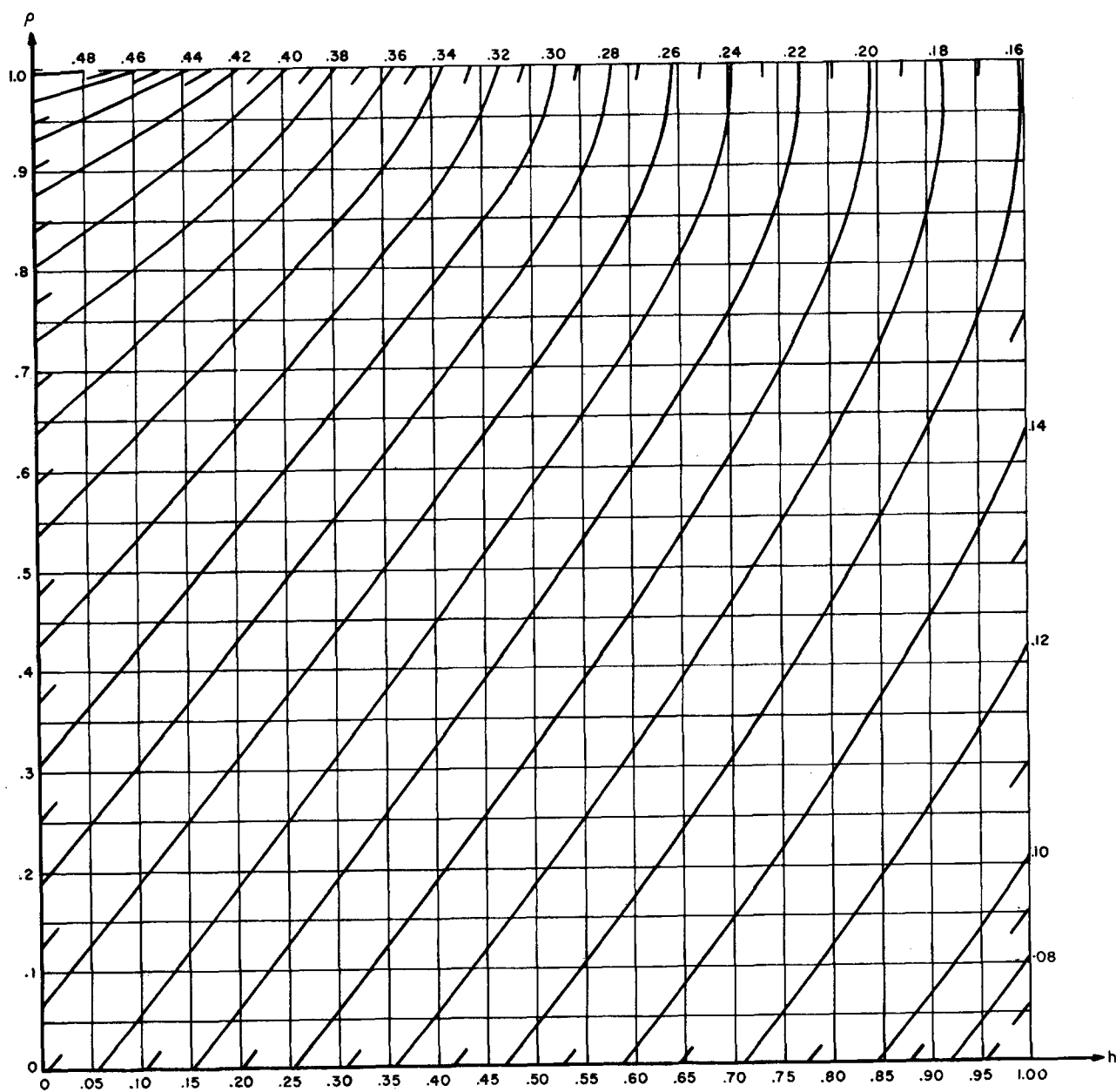


FIGURE 26.3. $L(h, 0, \rho)$ for $0 \leq h \leq 1$ and $0 \leq \rho \leq 1$.

Values for $h < 0$ can be obtained using $L(h, 0, -\rho) = \frac{1}{2} - L(-h, 0, \rho)$.

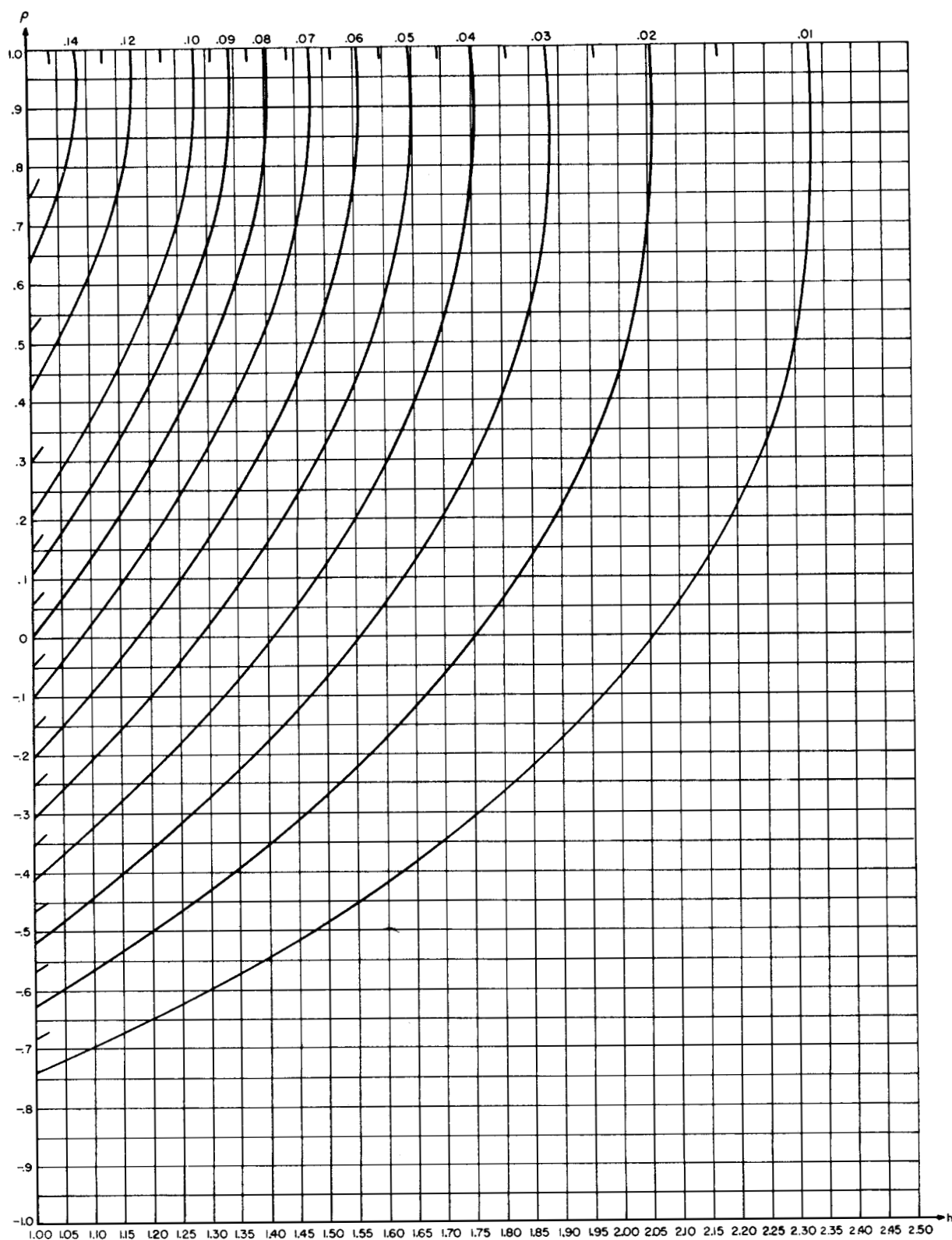


FIGURE 26.4. $L(h, 0, \rho)$ for $h \geq 1$ and $-1 \leq \rho \leq 1$.

Values for $h < 0$ can be obtained using $L(h, 0, -\rho) = \frac{1}{2} - L(-h, 0, \rho)$

Integral Over an Ellipse With Center at (m_x, m_y)

26.3.21

$$\iint_A (\sigma_x \sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dx dy = 1 - e^{-a^2/2}$$

where A is the area enclosed by the ellipse

$$\left(\frac{x-m_x}{\sigma_x}\right)^2 - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x \sigma_y} + \left(\frac{y-m_y}{\sigma_y}\right)^2 = a^2(1-\rho^2)$$

Integral Over an Arbitrary Region

26.3.22

$$\begin{aligned} \iint_{A(x,y)} (\sigma_x \sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dx dy \\ = \iint_{A^*(s,t)} g(s, t, \rho) ds dt \end{aligned}$$

where $A^*(s, t)$ is the transformed region obtained from the transformation

$$s = \frac{1}{\sqrt{2+2\rho}} \left(\frac{x-m_x}{\sigma_x} + \frac{y-m_y}{\sigma_y} \right)$$

$$t = \frac{-1}{\sqrt{2-2\rho}} \left(\frac{x-m_x}{\sigma_x} - \frac{y-m_y}{\sigma_y} \right)$$

Integral of the Circular Normal Probability Function With Parameters $m_x=m_y=0$, $\sigma=1$ Over the Triangle Bounded by $y=0$, $y=ax$, $x=h$

26.3.23

$$\begin{aligned} V(h, ah) &= \frac{1}{2\pi} \int_0^h \int_0^{ax} e^{-\frac{1}{2}(x^2+y^2)} dx dy \\ &= \frac{1}{4} + L(h, 0, \rho) - L(0, 0, \rho) - \frac{1}{2} Q(h) \end{aligned}$$

where

$$\rho = -\frac{a}{\sqrt{1+a^2}}$$

Integral of Circular Normal Distribution Over an Offset Circle With Radius $R\sigma$ and Center a Distance $r\sigma$ From (m_x, m_y)

26.3.24

$$\int_A \int \sigma^{-2} g\left(\frac{x-m_x}{\sigma}, \frac{y-m_y}{\sigma}, 0\right) dx dy = P(R^2|2, r^2)$$

where $P(R^2|2, r^2)$ is the c.d.f. of the non-central χ^2 distribution (see 26.4.25) with $\nu=2$ degrees of freedom and noncentrality parameter r^2 .

Approximation to $P(R^2|2, r^2)$

26.3.25

Approximation	Condition
$\frac{2R^2}{4+R^2} \exp -\frac{2r^2}{4+R^2}$	$R < 1$

26.3.26 $P(x_1)$	$R > 1$
------------------	---------

26.3.27 $P(x_2)$	$R > 5$
------------------	---------

$$x_1 = \frac{[R^2/(2+r^2)]^{1/3} - \left[1 - \frac{2}{9} \frac{2+2r^2}{(2+r^2)^2}\right]}{\left[\frac{2}{9} \frac{2+2r^2}{(2+r^2)^2}\right]^{1/2}}$$

$$x_2 = R - \sqrt{r^2 - 1} \quad R, r \text{ both large} \quad *$$

Inequality

26.3.28

$$Q(h) - \frac{1-\rho^2}{\rho h - k} Z(k) \left[Q\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right) \right] < L(h, k, \rho) < Q(h)$$

where

$$\rho h - k > 0, \quad 0 < \rho < 1.$$

Series Expansion

26.3.29

$$L(h, k, \rho) = Q(h) Q(k) + \sum_{n=0}^{\infty} \frac{Z^{(n)}(h) Z^{(n)}(k)}{(n+1)!} \rho^{n+1}$$

26.4. Chi-Square Probability Function

26.4.1

$$P(\chi^2|\nu) = \left[2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right) \right]^{-1} \int_0^{\chi^2} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \quad (0 \leq \chi^2 < \infty)$$

26.4.2

$$\begin{aligned} Q(\chi^2|\nu) &= 1 - P(\chi^2|\nu) \\ &= \left[2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right) \right]^{-1} \int_{\chi^2}^{\infty} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \end{aligned} \quad (0 \leq \chi^2 < \infty)$$

Relation to Normal Distribution

Let X_1, X_2, \dots, X_ν be independent and identically distributed random variables each following a normal distribution with mean zero and unit variance. Then $X^2 = \sum_{i=1}^{\nu} X_i^2$ is said to follow the chi-square distribution with ν degrees of freedom and the probability that $X^2 \leq \chi^2$ is given by $P(\chi^2|\nu)$.

Cumulants

$$26.4.3 \quad \kappa_{n+1} = 2^n n! \nu \quad (n=0, 1, \dots)$$

Series Expansions

26.4.4

$$Q(\chi^2|\nu) = 2Q(\chi) + 2Z(\chi) \sum_{r=1}^{\frac{\nu-1}{2}} \frac{\chi^{2r-1}}{1 \cdot 3 \cdot 5 \dots (2r-1)}$$

 $(\nu \text{ odd}) \text{ and } \chi = \sqrt{\chi^2}$

26.4.5

$$Q(\chi^2|\nu) = \sqrt{2\pi} Z(\chi) \left\{ 1 + \sum_{r=1}^{\frac{\nu-2}{2}} \frac{\chi^{2r}}{2 \cdot 4 \dots (2r)} \right\}$$

$(\nu \text{ even})$

26.4.6

$$P(\chi^2|\nu) = \left(\frac{1}{2}\chi^2\right)^{\nu/2} \frac{e^{-\chi^2/2}}{\Gamma\left(\frac{\nu+2}{2}\right)}$$

$$* \left\{ 1 + \sum_{r=1}^{\infty} \frac{\chi^{2r}}{(\nu+2)(\nu+4)\dots(\nu+2r)} \right\}$$

$$26.4.7 \quad P(\chi^2|\nu) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \sum_{n=0}^{\infty} \frac{(-1)^n (\chi^2/2)^{\frac{\nu}{2}+n}}{n! \left(\frac{\nu}{2}+n\right)}$$

Recurrence and Differential Relations

$$26.4.8 \quad Q(\chi^2|\nu+2) = Q(\chi^2|\nu) + \frac{(\chi^2/2)^{\nu/2} e^{-\chi^2/2}}{\Gamma\left(\frac{\nu}{2}+1\right)}$$

$$26.4.9 \quad \frac{\partial^m Q(\chi^2|\nu)}{\partial (\chi^2)^m} = \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} (-1)^{m+j} Q(\chi^2|\nu-2j)$$

Continued Fraction

$$26.4.10 \quad *Q(\chi^2|\nu) = \frac{(\chi^2)^{\nu/2} e^{-\chi^2/2}}{2^{\nu/2} \Gamma(\nu/2)}$$

$$\left\{ \frac{1}{\chi^2/2+} \frac{1-\nu/2}{1+} \frac{1}{\chi^2/2+} \frac{2-\nu/2}{1+} \frac{2}{\chi^2/2+} \dots \right\}$$

Asymptotic Distribution for Large ν

$$26.4.11 \quad P(\chi^2|\nu) \sim P(x) \quad \text{where } x = \frac{\chi^2 - \nu}{\sqrt{2\nu}}$$

Asymptotic Expansions for Large χ^2

26.4.12

$$Q(\chi^2|\nu) \sim \frac{(\chi^2)^{\frac{\nu}{2}-1} e^{-\chi^2/2}}{2^{\nu/2} \Gamma(\nu/2)} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma\left(1-\frac{\nu}{2}+j\right)}{\Gamma\left(1-\frac{\nu}{2}\right)} \frac{2^{j+1}}{(\chi^2)^j}$$

*See page II.

Approximations to the Chi-Square Distribution for Large ν

26.4.13

$$Q(\chi^2|\nu) \approx Q(x_1), \quad x_1 = \sqrt{2\chi^2} - \sqrt{2\nu-1} \quad (\nu > 100)$$

Approximation *Condition*

26.4.14

$$Q(\chi^2|\nu) \approx Q(x_2), \quad x_2 = \frac{(\chi^2/\nu)^{1/3} - \left(1 - \frac{2}{9\nu}\right)}{\sqrt{2/9\nu}} \quad (\nu > 30)$$

26.4.15

$$Q(\chi^2|\nu) \approx Q(x_2 + h_\nu), \quad h_\nu = \frac{60}{\nu} h_{80} \quad (\nu > 30)$$

Values of h_{80}

x	h_{80}	x	h_{80}	x	h_{80}
-3.5	-.0118	-1.0	+.0006	+1.5	-.0005
-3.0	-.0067	-.5	+.0006	2.0	+.0002
-2.5	-.0033	0	+.0002	2.5	.0017
-2.0	-.0010	+.5	-.0003	3.0	.0043
-1.5	+.0001	1.0	-.0006	3.5	.0082

Approximations for the Inverse Function for Large ν If $Q(\chi_p^2|\nu) = p$ and $Q(x_p) = 1 - P(x_p) = p$, then

$$26.4.16 \quad \chi_p^2 \approx \frac{1}{2} \left\{ x_p + \sqrt{2\nu-1} \right\}^2 \quad (\nu > 100)$$

Approximation *Condition*

$$26.4.17 \quad \chi_p^2 \approx \nu \left\{ 1 - \frac{2}{9\nu} + x_p \sqrt{\frac{2}{9\nu}} \right\}^3 \quad (\nu > 30)$$

$$26.4.18 \quad \chi_p^2 \approx \nu \left\{ 1 - \frac{2}{9\nu} + (x_p - h_\nu) \sqrt{\frac{2}{9\nu}} \right\}^3 \quad (\nu > 30)$$

where h_ν is given by 26.4.15.

Relation to Other Functions

26.4.19 Incomplete gamma function

$$\frac{\gamma(a, x)}{\Gamma(a)} = P(\chi^2|\nu), \quad \nu = 2a, \chi^2 = 2x$$

$$\frac{\Gamma(a, x)}{\Gamma(a)} = Q(\chi^2|\nu)$$

26.4.20 Pearson's incomplete gamma function

$$I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} t^p e^{-t} dt = P(\chi^2|\nu)$$

$$\nu = 2(p+1), \chi^2 = 2u\sqrt{p+1}$$

26.4.21 Poisson distribution

$$Q(\chi^2|\nu) = \sum_{j=0}^{c-1} e^{-m} \frac{m^j}{j!}, \quad c = \frac{\nu}{2}, m = \frac{\chi^2}{2}, (\nu \text{ even})$$

$$Q(\chi^2|\nu) - Q(\chi^2|\nu-2) = e^{-m} \frac{m^{c-1}}{(c-1)!}$$

26.4.22 Pearson Type III

$$\left[\frac{ab}{e}\right] \int_{-a}^x \left(1 + \frac{t}{a}\right)^{ab} e^{-bt} dt = P(\chi^2|\nu)$$

$$\nu = 2ab + 2, \chi^2 = 2b(x + a)$$

26.4.23 Incomplete moments of Normal distribution

$$\int_0^x t^n Z(t) dt = \begin{cases} (n-1)!! \frac{P(\chi^2|\nu)}{2} & (n \text{ even}) \\ \frac{(n-1)!!}{\sqrt{2\pi}} P(\chi^2|\nu) & (n \text{ odd}) \end{cases}$$

$$\chi^2 = x^2, \nu = n + 1$$

26.4.24 Generalized Laguerre Polynomials

$$n! L_n^{(\alpha)}(x) = \frac{\sum_{j=0}^{n+1} (-1)^{n+j} \binom{n+1}{j} Q(\chi^2|\nu+2-2j)}{2^n [Q(\chi^2|\nu+2) - Q(\chi^2|\nu)]}$$

$$x = \chi^2/2, \alpha = \nu/2$$

Non-Central χ^2 Distribution Function

26.4.25

$$P(\chi'^2|\nu, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} P(\chi'^2|\nu+2j)$$

where $\lambda \geq 0$ is termed the non-centrality parameter.

Relation of Non-Central χ^2 Distribution With $\nu=2$ to the Integral of Circular Normal Distribution ($\sigma^2=1$) Over an Offset Circle Having Radius R and Center a Distance $r=\sqrt{\lambda}$ From the Origin. (See 26.3.24-26.3.27.)

26.4.26

$$\iint_A g(x, y, 0) dx dy = P(\chi^2 = R^2|\nu=2, \lambda)$$

$$= 1 - \sum_{j=0}^{\infty} \frac{e^{-\lambda/2} \lambda^j}{2^j j!} Q(R^2|2+2j)$$

Approximations to the Non-Central χ^2 Distribution

$$a = \nu + \lambda \quad b = \frac{\lambda}{\nu + \lambda}$$

Approximating Function

Approximation

26.4.27 χ^2 distribution $P(\chi'^2|\nu, \lambda) \approx P\left(\frac{\chi^2}{1+b}|\nu^*\right), \quad \nu^* = \frac{a}{1+b}$

26.4.28 Normal distribution $P(\chi'^2|\nu, \lambda) \approx P(x), \quad x = \frac{(\chi'^2/a)^{1/3} - \left[1 - \frac{2}{9}\left(\frac{1+b}{a}\right)\right]}{\sqrt{\frac{2}{9}\left(\frac{1+b}{a}\right)}}$

26.4.29 Normal distribution $P(\chi'^2|\nu, \lambda) \approx P(x), \quad x = \left[\frac{2\chi'^2}{1+b}\right]^{1/2} - \left[\frac{2a}{1+b} - 1\right]^{1/2}$

Approximations to the Inverse Function of Non-Central χ^2 Distribution

If $Q(\chi_p'^2|\nu, \lambda) = p$, $Q(\chi_p^2|\nu^*) = p$, and $Q(x_p) = p$ then

Approximating Variable

Approximation to the Inverse Function

26.4.30 $\chi^2 \quad \chi_p'^2 \approx (1+b)\chi_p^2$

26.4.31 Normal $\chi_p'^2 \approx \frac{1+b}{2} \left[x_p + \sqrt{\frac{2a}{1+b} - 1} \right]^2$

26.4.32 Normal $\chi_p'^2 \approx a \left[x_p \sqrt{\frac{2(1+b)}{a}} + 1 - \frac{2}{9} \left(\frac{1+b}{a} \right) \right]^2$

Properties of Chi-Square, Non-Central Chi-Square, and Related Quantities

$$a = r + \lambda \quad b = \frac{\lambda}{r + \lambda}$$

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z), \quad \psi^*(z) = \frac{d^2}{dz^2} \psi(z)$$

Variable	Mean	Variance	Coefficient of skewness (γ_1)	Coefficient of excess (γ_2)
26.4.33 χ^2	r	$2r$	$\frac{2\sqrt{r}}{\sqrt{r}}$	$12r^{-1}$
26.4.34 $\sqrt{2\chi^2}$	$(2r-1)^{\frac{1}{2}} \{1 + [16r(r-1)]^{-1}\} + O(r^{-1/2})$	$1 - \frac{1}{4r} - \frac{1}{8r^2} + \frac{5}{64r^3} - O(r^{-4})$	$\frac{1}{\sqrt{2r}} \left[1 + \frac{5}{8r} - \frac{1}{128r^2} \right] + O(r^{-1/2})$	$\frac{3}{2^{\frac{3}{2}}} \frac{1}{r^{\frac{3}{2}}} \left[1 + \frac{3}{2r} \right] + O(r^{-2})$
26.4.35 $(\chi^2/r)^{1/2}$	$1 - \frac{2}{3r} + \frac{80}{3^2 r^2} + O(r^{-3})$	$\frac{2}{3r} - \frac{104}{3^2 r^2} + O(r^{-3})$	$\frac{2\sqrt{r}}{3^{\frac{3}{2}} r^{\frac{3}{2}}} \left[1 + \frac{8}{3r} \right] + O(r^{-1/2})$	$-\frac{4}{9r} \left[1 + \frac{16}{9r} \right] + O(r^{-2})$
26.4.36 $\ln(\chi^2/r)$	$\psi\left(\frac{r}{2}\right) - \ln\left(\frac{r}{2}\right) = -\frac{1}{r} - \frac{1}{3r^2} + O(r^{-3})$	$\psi'\left(\frac{r}{2}\right) = \frac{2}{r-1} \left[1 - \frac{1}{3(r-1)^2} \right] + O((r-1)^{-3})$	$\frac{\psi''\left(\frac{r}{2}\right)}{\psi'\left(\frac{r}{2}\right)^{3/2}} = -\sqrt{\frac{2}{r-1}} \left[1 - \frac{1}{2(r-1)^2} \right] + O((r-1)^{-3/2})$	$\frac{\psi^{(3)}\left(\frac{r}{2}\right)}{\psi'\left(\frac{r}{2}\right)^{\frac{5}{2}}} = \frac{4}{r-1} \left[1 + \frac{4}{3(r-1)^2} \right] + O((r-1)^{-3})$
26.4.37 χ'^2	a	$2a(1+b)$	$\left(\frac{2}{1+b}\right)^{3/2} (1+2b)a^{-\frac{1}{2}}$	$\frac{12}{a} \frac{(1+3b)}{(1+b)^3}$
26.4.38 $\sqrt{2\chi'^2}$	$[2a - (1+b)]^{\frac{1}{2}} + O(a^{-3/2})$	$(1+b) - \frac{a^{-1}}{4} [8b + (1+b)(1-7b)] + O(a^{-2})$	$\frac{a^{-\frac{1}{2}}(1-b)(1+3b)}{2^{\frac{3}{2}}(1+b)^{3/2}} + O(a^{-1})$	$\frac{3b(b+2)}{(1+b)^3} + O(a^{-2})$
26.4.39 $(\chi'^2/a)^{1/2}$	$1 - \frac{2}{3^2} \frac{1+b}{a} - \frac{40}{3^4} \frac{b^2}{a^2} + O(a^{-3})$	$\frac{2}{9} a^{-1}(1+b) + \frac{16}{27} a^{-2}b + O(a^{-3})$	$\left(\frac{2}{1+b}\right)^{3/2} b a^{-\frac{1}{2}} + O(a^{-3/2})$	$-\frac{4}{3^3} \frac{(1+3b+12b^2-44b^3)}{a(1+b)^3} - O(a^{-2})$

26.5. Incomplete Beta Function

26.5.1

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt \quad (0 \leq x \leq 1)$$

26.5.2

$$I_x(a, b) = 1 - I_{1-x}(b, a)$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions 26.4.1 with ν_1 and ν_2 degrees of freedom respectively, then $\frac{X_1^2}{X_1^2 + X_2^2}$ is said to follow a beta distribution with ν_1 and ν_2 degrees of freedom and has the distribution function

26.5.3

$$P\left\{\frac{X_1^2}{X_1^2 + X_2^2} \leq x\right\} = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt \\ = I_x(a, b) \quad a = \frac{\nu_1}{2}, b = \frac{\nu_2}{2}$$

Series Expansions ($0 < x < 1$)

26.5.4

$$* I_x(a, b) = \frac{x^a (1-x)^b}{a B(a, b)} \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(a+b, n+1)} x^{n+1} \right\}$$

26.5.5

$$I_x(a, b) = \frac{x^a (1-x)^{b-1}}{a B(a, b)} \\ \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(b-n-1, n+1)} \left(\frac{x}{1-x}\right)^{n+1} \right\} \\ = \frac{x^a (1-x)^{b-1}}{a B(a, b)} \\ \left\{ 1 + \sum_{n=0}^{b-2} \frac{B(a+1, n+1)}{B(b-n-1, n+1)} \left(\frac{x}{1-x}\right)^{n+1} \right\} \\ + I_x(a+s, b-s)$$

26.5.6

$$1 - I_x(a, b) = I_{1-x}(b, a) \\ = \frac{(1-x)^b}{B(a, b)} \sum_{i=0}^{a-1} (-1)^i \binom{a-1}{i} \frac{(1-x)^i}{b+i} \quad (\text{integer } a)$$

26.5.7

$$1 - I_x(a, b) = I_{1-x}(b, a) \\ = (1-x)^{a+b-1} \sum_{i=0}^{a-1} \binom{a+b-1}{i} \left(\frac{x}{1-x}\right)^i \quad (\text{integer } a)$$

Continued Fractions

26.5.8

$$I_x(a, b) = \frac{x^a (1-x)^b}{a B(a, b)} \left\{ \frac{1}{1+} \frac{d_1}{1+} \frac{d_2}{1+} \dots \right\} *$$

$$d_{2m+1} = -\frac{(a+m)(a+b+m)}{(a+2m)(a+2m+1)} x$$

$$d_{2m} = \frac{m(b-m)}{(a+2m-1)(a+2m)} x$$

Best results are obtained when $x < \frac{a-1}{a+b-2}$.

Also the $4m$ and $4m+1$ convergents are less than $I_x(a, b)$ and the $4m+2$, $4m+3$ convergents are greater than $I_x(a, b)$.

26.5.9

$$I_x(a, b) = \frac{x^a (1-x)^{b-1}}{a B(a, b)} \left[\frac{e_1}{1+} \frac{e_2}{1+} \frac{e_3}{1+} \dots \right]$$

$$* \quad x < 1 \quad e_1 = 1$$

$$e_{2m} = -\frac{(a+m-1)(b-m)}{(a+2m-2)(a+2m-1)} \frac{x}{1-x}$$

$$e_{2m+1} = \frac{m(a+b-1+m)}{(a+2m-1)(a+2m)} \frac{x}{1-x}$$

Recurrence Relations

26.5.10

$$I_x(a, b) = x I_x(a-1, b) + (1-x) I_x(a, b-1)$$

26.5.11

$$I_x(a, b) = \frac{1}{x} \{ I_x(a+1, b) - (1-x) I_x(a+1, b-1) \}$$

26.5.12

$$[I_x(a, b)] = \frac{1}{a(1-x)+b} \{ b I_x(a, b+1) + a(1-x) I_x(a+1, b-1) \} *$$

26.5.13

$$I_x(a, b) = \frac{1}{a+b} \{ a I_x(a+1, b) + b I_x(a, b+1) \}$$

26.5.14

$$I_x(a, a) = \frac{1}{2} I_{1-x'}\left(a, \frac{1}{2}\right), \quad x' = 4\left(x - \frac{1}{2}\right)^2 \left[x \leq \frac{1}{2} \right]$$

26.5.15

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-x)^{b-1} + I_x(a+1, b-1)$$

26.5.16

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-x)^b + I_x(a+1, b)$$

Asymptotic Expansions

26.5.17

$$1 - I_x(a, b) = I_{1-x}(b, a) \sim \frac{\Gamma(b, y)}{\Gamma(b)}$$

$$- \frac{1}{24N^2} \left\{ \frac{y^b e^{-y}}{(b-2)!} (b+1+y) \right\}$$

$$+ \frac{1}{5760N^4} \left\{ \frac{y^b e^{-y}}{(b-2)!} [(b-3)(b-2)(5b+7)(b+1+y) \right.$$

$$\left. - (5b-7)(b+3+y)y^2] \right\}$$

$$y = -N \ln x, \quad N = a + \frac{b}{2} - \frac{1}{2}$$

26.5.18

$$I_x(a, b) \sim \frac{\Gamma(a, w)}{\Gamma(a)} + \frac{e^{-w} w^a}{\Gamma(a)} \left\{ \frac{(a-1-w)}{2b} \right.$$

$$+ \frac{1}{(2b)^2} \left(\frac{a^3}{2} - \frac{5}{3} a^2 + \frac{3}{2} a - \frac{1}{3} - w \left[\frac{3}{2} a^2 - \frac{11}{6} a + \frac{1}{3} \right] \right.$$

$$\left. \left. + w^2 \left(\frac{3}{2} a - \frac{1}{6} \right) - \frac{1}{2} w^3 \right) \right\}$$

$$w = b \left(\frac{x}{1-x} \right)$$

26.5.19

$$I_x(a, b) \sim P(y) - Z(y) \left[a_1 + \frac{a_2(y-a_1)}{1+a_2} \right.$$

$$\left. + \frac{a_3(1+y^2/2)}{1+a_2} + \dots \right]$$

$$a_1 = \frac{2}{3} (b-a) [(a+b-2)(a-1)(b-1)]^{-1/2}$$

$$a_2 = \frac{1}{12} \left[\frac{1}{a-1} + \frac{1}{b-1} - \frac{13}{a+b-1} \right]$$

$$a_3 = -\frac{8}{15} \left[a_1 \left(a_2 + \frac{3}{a+b-2} \right) \right]$$

$$y^2 = 2 \left[(a+b-1) \ln \frac{a+b-1}{a+b-2} + (a-1) \ln \frac{a-1}{(a+b-1)x} \right.$$

$$\left. + (b-1) \ln \frac{b-1}{(a+b-1)(1-x)} \right]$$

and y is taken negative when $x < \frac{a-1}{a+b-2}$

Approximations

26.5.20 If $(a+b-1)(1-x) \leq .8$

$$I_x(a, b) = Q(x^2 | \nu) + \epsilon,$$

$$|\epsilon| < 5 \times 10^{-3} \text{ if } a+b > 6$$

$$x^2 = (a+b-1)(1-x)(3-x) - (1-x)(b-1),$$

$$\nu = 2b$$

26.5.21 If $(a+b-1)(1-x) \geq .8$

$$I_x(a, b) = P(y) + \epsilon,$$

$$|\epsilon| < 5 \times 10^{-3} \text{ if } a+b > 6$$

$$y = \frac{3 \left[w_1 \left(1 - \frac{1}{9b} \right) - w_2 \left(1 - \frac{1}{9a} \right) \right]}{\left[\frac{w_1^2}{b} + \frac{w_2^2}{a} \right]^{1/2}},$$

$$w_1 = (bx)^{1/3}, w_2 = [a(1-x)]^{1/3}$$

Approximation to the Inverse Function

26.5.22 If $I_{x_p}(a, b) = p$ and $Q(y_p) = p$ then

$$x_p \approx \frac{a}{a + b e^{2w}}$$

$$w = \frac{y_p(h+\lambda)^{1/2}}{h} - \left(\frac{1}{2b-1} - \frac{1}{2a-1} \right) \left(\lambda + \frac{5}{6} - \frac{2}{3h} \right)$$

$$h = 2 \left(\frac{1}{2a-1} + \frac{1}{2b-1} \right)^{-1}, \quad \lambda = \frac{y_p^2 - 3}{6}$$

Relations to Other Functions and Distributions

Function

Relation

26.5.23 Hypergeometric function

$$\frac{1}{B(a, b)} \frac{x^a}{a} F(a, 1-b; a+1; x) = I_x(a, b)$$

26.5.24 Binomial distribution

$$\sum_{s=0}^n \binom{n}{s} p^s (1-p)^{n-s} = I_p(a, n-a+1)$$

26.5.25

"

$$\binom{n}{a} p^a (1-p)^{n-a} = I_p(a, n-a+1) - I_p(a+1, n-a) *$$

26.5.26 Negative binomial distribution

$$\sum_{s=0}^n \binom{n+s-1}{s} p^n q^s = I_q(a, n)$$

26.5.27 Student's distribution

$$\frac{1}{2} [1 - A(t|\nu)] = \frac{1}{2} I_x \left(\frac{\nu}{2}, \frac{1}{2} \right), \quad x = \frac{\nu}{\nu + t^2} *$$

26.5.28 F -(variance-ratio) distribution

$$Q(F|\nu_1, \nu_2) = I_x \left(\frac{\nu_2}{2}, \frac{\nu_1}{2} \right), \quad x = \frac{\nu_2}{\nu_2 + \nu_1 F}$$

* See page 11.

26.6. F-(Variance-Ratio) Distribution Function**26.6.1**

$$P(F|v_1, v_2) = \frac{v_1^{v_1} v_2^{v_2}}{B\left(\frac{1}{2}v_1, \frac{1}{2}v_2\right)} \int_0^F t^{\frac{1}{2}(v_1-2)} (v_2+v_1 t)^{-\frac{1}{2}(v_1+v_2)} dt \quad (F \geq 0)$$

26.6.2

$$Q(F|v_1, v_2) = 1 - P(F|v_1, v_2) = I_x\left(\frac{v_2}{2}, \frac{v_1}{2}\right)$$

where

$$x = \frac{v_2}{v_2 + v_1 F}$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions 26.4.1 with v_1 and v_2 degrees of freedom respectively, then the distribution of $F = \frac{X_1^2/v_1}{X_2^2/v_2}$ is said to follow the variance ratio or *F*-distribution with v_1 and v_2 degrees of freedom. The corresponding distribution function is $P(F|v_1, v_2)$.

Statistical Properties**26.6.3**

mean: $m = \frac{v_2}{v_2 - 2} \quad (v_2 > 2)$

variance: $\sigma^2 = \frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2 - 2)^2(v_2 - 4)} \quad (v_2 > 4)$

third central moment:

$$\mu_3 = \left(\frac{v_2}{v_1}\right)^3 \frac{8v_1(v_1 + v_2 - 2)(2v_1 + v_2 - 2)}{(v_2 - 2)^3(v_2 - 4)(v_2 - 6)} \quad (v_2 > 6)$$

moments about the origin:

$$\mu'_n = \left(\frac{v_2}{v_1}\right)^n \frac{\Gamma\left(\frac{v_1 + 2n}{2}\right) \Gamma\left(\frac{v_1 - 2n}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)} \quad (v_2 > 2n)$$

characteristic function:

$$\phi(t) = E(e^{itF}) = M\left(\frac{v_1}{2}, -\frac{v_2}{2}, -\frac{v_2}{v_1} it\right)$$

Series Expansions

$$x = \frac{v_2}{v_2 + v_1 F}$$

26.6.4

$$* Q(F|v_1, v_2) = x^{v_2/2} \left[1 + \frac{v_2}{2}(1-x) + \frac{v_2(v_2+2)}{2 \cdot 4}(1-x)^2 + \dots + \frac{v_2(v_2+2) \dots (v_2+v_1-4)}{2 \cdot 4 \dots (v_1-2)} (1-x)^{\frac{v_1-2}{2}} \right] \quad (v_1 \text{ even})$$

26.6.5

$$Q(F|v_1, v_2) = 1 - (1-x)^{v_1/2} \left[1 + \frac{v_1}{2}x + \frac{v_1(v_1+2)}{2 \cdot 4}x^2 + \dots + \frac{v_1(v_1+2) \dots (v_2+v_1-4)}{2 \cdot 4 \dots (v_2-2)} x^{\frac{v_2-2}{2}} \right] \quad (v_2 \text{ even})$$

26.6.6

$$Q(F|v_1, v_2) = x^{\frac{v_1+v_2-2}{2}} \left[1 + \frac{v_1+v_2-2}{2} \left(\frac{1-x}{x}\right) + \frac{(v_1+v_2-2)(v_1+v_2-4)}{2 \cdot 4} \left(\frac{1-x}{x}\right)^2 + \dots + \frac{(v_1+v_2-2) \dots (v_2+2)}{2 \cdot 4 \dots (v_1-2)} \left(\frac{1-x}{x}\right)^{\frac{v_1-2}{2}} \right] \quad (v_1 \text{ even})$$

26.6.7

$$Q(F|v_1, v_2) = 1 - (1-x)^{\frac{v_1+v_2-2}{2}} \left[1 + \frac{v_1+v_2-2}{2} \left(\frac{x}{1-x}\right) + \dots + \frac{(v_1+v_2-2) \dots (v_1+2)}{2 \cdot 4 \dots (v_2-2)} \left(\frac{x}{1-x}\right)^{\frac{v_2-2}{2}} \right] \quad (v_2 \text{ even})$$

26.6.8

$$Q(F|v_1, v_2) = 1 - A(t|v_2) + \beta(v_1, v_2) \quad (v_1, v_2 \text{ odd})$$

$$A(t|v_2) = \begin{cases} \frac{2}{\pi} \left\{ \theta + \sin \theta \left[\cos \theta + \frac{2}{3} \cos^3 \theta + \dots + \frac{2 \cdot 4 \dots (v_2-3)}{3 \cdot 5 \dots (v_2-2)} \cos^{v_2-2} \theta \right] \right\} & \text{for } v_2 > 1 \\ \frac{2\theta}{\pi} & \text{for } v_2 = 1 \end{cases}$$

$$\beta(v_1, v_2) = \begin{cases} \frac{2}{\sqrt{\pi}} \left(\frac{v_2-1}{2} \right)! \sin \theta \cos^{v_2} \theta \left\{ 1 + \frac{v_2+1}{3} \sin^2 \theta + \dots + \frac{(v_2+1)(v_2+3) \dots (v_1+v_2-4)}{3 \cdot 5 \dots (v_1-2)} \sin^{v_1-3} \theta \right\} & \text{for } v_2 > 1 \\ 0 & \text{for } v_1 = 1 \end{cases} *$$

where

$$\theta = \arctan \sqrt{\frac{v_1}{v_2} F}$$

Reflexive Relation

If $F_p(v_1, v_2)$ and $F_{1-p}(v_2, v_1)$ satisfy

$$Q(F_p(v_1, v_2)|v_1, v_2) = p$$

$$Q(F_{1-p}(v_2, v_1)|v_2, v_1) = 1 - p$$

26.6.9 then

$$F_p(\nu_1, \nu_2) = \frac{1}{F_{1-p}(\nu_2, \nu_1)}$$

Relation to Student's t -Distribution Function (See 26.7)

$$26.6.10 \quad Q(F|\nu_1=1, \nu_2)=1-A(t|\nu_2) \quad t=\sqrt{F}$$

Limiting Forms

26.6.11

$$\lim_{\nu_1 \rightarrow \infty} Q(F|\nu_1, \nu_2) = Q(\chi^2|\nu_1), \quad \chi^2 = \nu_1 F$$

26.6.12

$$\lim_{\nu_1 \rightarrow \infty} Q(F|\nu_1, \nu_2) = P(\chi^2|\nu_2), \quad \chi^2 = \frac{\nu_2}{F}$$

Approximations

26.6.13

$$Q(F|\nu_1, \nu_2) \approx Q(x), \quad x = \frac{F - \frac{\nu_2}{\nu_2 - 2}}{\frac{\nu_2}{\nu_2 - 2} \sqrt{\frac{2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 4)}}}$$

(ν_1 and ν_2 large)

26.6.14

$$Q(F|\nu_1, \nu_2) \approx Q(x), \quad x = \frac{\sqrt{(2\nu_2 - 1) \frac{\nu_1}{\nu_2} F - \sqrt{2\nu_1 - 1}}}{\sqrt{1 + \frac{\nu_1}{\nu_2} F}}$$

Relation of Non-Central F -Distribution Function to Other Functions

Function

26.6.18 F -distribution

26.6.19 Non-central t -distribution

26.6.20 Incomplete Beta function

26.6.21 Confluent hypergeometric function

26.6.15

$$Q(F|\nu_1, \nu_2) \approx Q(x), \quad x = \frac{F^{1/3} \left(1 - \frac{2}{9\nu_2}\right) - \left(1 - \frac{2}{9\nu_1}\right)}{\sqrt{\frac{2}{9\nu_1} + F^{2/3} \frac{2}{9\nu_2}}}$$

Approximation to the Inverse Function

26.6.16 If $Q(F_p|\nu_1, \nu_2) = p$, then

$$F_p \approx e^{2w} \text{ where } w \text{ is given by 26.5.22, with } \nu_1 = 2b, \nu_2 = 2a$$

Non-Central F -Distribution Function

26.6.17

$$P(F'|\nu_1, \nu_2, \lambda) = \int_0^{F'} p(t|\nu_1, \nu_2, \lambda) dt = 1 - Q(F'|\nu_1, \nu_2, \lambda)$$

where

$$p(t|\nu_1, \nu_2, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} \frac{\nu_1 + 2j}{B\left(\frac{\nu_1 + 2j}{2}, \frac{\nu_2}{2}\right)} \times t^{\frac{\nu_1 + 2j - 2}{2}} [\nu_2 + (\nu_1 + 2j)t]^{-(\nu_1 + 2j + \nu_2)/2}$$

and $\lambda \geq 0$ is termed the non-centrality parameter.

Relation

$$P(F'|\nu_1, \nu_2, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} P(F'|\nu_1 + 2j, \nu_2)$$

$$P(F'|\nu_1, \nu_2, \lambda=0) = P(F'|\nu_1, \nu_2)$$

$$P(F'|\nu_1=1, \nu_2, \lambda) = P(t'|\nu, \delta), \quad t' = \sqrt{F'}, \nu = \nu_2, \delta = \sqrt{\lambda}$$

$$P(F'|\nu_1, \nu_2) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} I_x\left(\frac{\nu_1}{2} + j, \frac{\nu_2}{2}\right),$$

$$x = \frac{\nu_1 F'}{\nu_1 F' + \nu_2} *$$

$$P(F'|\nu_1, \nu_2, \lambda) = \sum_{i=0}^{\frac{\nu_2}{2}-1} \frac{2e^{-\lambda/2}}{(\nu_1 + \nu_2) B\left(\frac{\nu_1}{2} + i + 1, \frac{\nu_2}{2} - i\right)} \times$$

$$x^{\frac{\nu_1}{2}+1} (1-x)^{\frac{\nu_2}{2}-i-1} M\left(\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1}{2} + i + 1, \frac{\lambda x}{2}\right)$$

$$(\nu_2 \text{ even and } x = \frac{\nu_2}{\nu_1 F' + \nu_2})$$

Series Expansion

26.6.22

$$P(F'|v_1, v_2, \lambda) = e^{-\frac{\lambda}{2}(1-x)} x^{\frac{1}{2}(v_1+v_2-2)} \sum_{i=0}^{\frac{v_2-1}{2}} T_i \quad (v_2 \text{ even})$$

where

$$T_0 = 1$$

$$T_1 = \frac{1}{2} (v_1 + v_2 - 2 + \lambda x) \frac{1-x}{x}$$

$$T_i = \frac{1-x}{2i} [(v_1 + v_2 - 2i + \lambda x) T_{i-1} + \lambda(1-x) T_{i-2}]$$

$$x = \frac{v_2}{v_1 F' + v_2}$$

Limiting Forms

26.6.23

$$\lim_{v_1 \rightarrow \infty} P(F'|v_1, v_2, \lambda) = P(\chi'^2|v, \lambda), \quad \chi'^2 = v_1 F', \quad v = v_1$$

26.6.24

$$\lim_{v_1 \rightarrow \infty} P(F'|v_1, v_2, \lambda) = Q(\chi^2|v), \quad \chi^2 = \frac{v_2(1+c^2)}{F'}$$

where $\lambda/v_1 \rightarrow c^2$ as $v_1 \rightarrow \infty$.

Approximations to the Non-Central F-Distribution

$$26.6.25 \quad P(F'|v_1, v_2, \lambda) \approx P(x_1), \quad (v_1 \text{ and } v_2 \text{ large})$$

where

$$x_1 = \frac{\frac{F' - \frac{v_2(v_1+\lambda)}{v_1(v_2-2)}}{\frac{v_2}{v_1} \left[\frac{2}{(v_2-2)(v_2-4)} \left\{ \frac{(v_1+\lambda)^2}{v_2-2} + v_1 + 2\lambda \right\} \right]^{\frac{1}{2}}}}$$

26.6.26

$$P(F'|v_1, v_2, \lambda) \approx P(F|v_1^*, v_2),$$

$$F = \frac{v_1}{v_1 + \lambda} F', \quad v_1^* = \frac{(v_1 + \lambda)^2}{v_1 + 2\lambda}$$

26.6.27

$$P(F'|v_1, v_2, \lambda) \approx P(x_2),$$

$$x_2 = \frac{\left[\frac{v_1 F'}{(v_1 + \lambda)} \right]^{1/3} \left[1 - \frac{2}{9v_2} \right] - \left[1 - \frac{2(v_1 + 2\lambda)}{9(v_1 + \lambda)^2} \right]}{\left[\frac{2}{9} \frac{v_1 + 2\lambda}{(v_1 + \lambda)^2} + \frac{2}{9v_2} \left(\frac{v_1}{v_1 + \lambda} F' \right)^{2/3} \right]^{\frac{1}{2}}}}$$

26.7. Student's *t*-Distribution

If X is a random variable following a normal distribution with mean zero and variance unity, and χ^2 is a random variable following an independent chi-square distribution with ν degrees of freedom, then the distribution of the ratio $\frac{X}{\sqrt{\chi^2/\nu}}$

is called Student's *t*-distribution with ν degrees of freedom. The probability that $\frac{X}{\sqrt{\chi^2/\nu}}$ will be less in absolute value than a fixed constant t is

26.7.1

$$A(t|\nu) = P, \left\{ \left| \frac{X}{\sqrt{\chi^2/\nu}} \right| \leq t \right\} \\ = \left[\sqrt{\nu} B\left(\frac{1}{2}, \frac{\nu}{2}\right) \right]^{-1} \int_{-t}^t \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\ = 1 - I_x\left(\frac{\nu}{2}, \frac{1}{2}\right), \quad (0 \leq t < \infty) *$$

where

$$x = \frac{\nu}{\nu + t^2}$$

Statistical Properties

26.7.2

$$\text{mean: } m = 0$$

$$\text{variance: } \sigma^2 = \frac{\nu}{\nu - 2} \quad (\nu > 2)$$

$$\text{skewness: } \gamma_1 = 0$$

$$\text{excess: } \gamma_2 = \frac{6}{\nu - 4} \quad (\nu > 4)$$

moments:

$$\mu_{2n} = \frac{1 \cdot 3 \cdot \dots \cdot (2n-1) \nu^n}{(\nu-2)(\nu-4) \cdot \dots \cdot (\nu-2n)} \quad (\nu > 2n)$$

$$\mu_{2n+1} = 0$$

characteristic function:

$$\phi(t) = E \left[\exp \left(it \frac{X}{\sqrt{\chi^2/\nu}} \right) \right] = \frac{\left(\frac{|t|}{2\sqrt{\nu}} \right)^{\nu/2}}{\pi \Gamma(\nu/2)} Y_{\frac{\nu}{2}} \left(\frac{|t|}{\sqrt{\nu}} \right)$$

Series Expansions

$$\left(\theta = \arctan \frac{t}{\sqrt{\nu}} \right)$$

26.7.3

$$A(t|\nu) = \begin{cases} \frac{2}{\pi} \left\{ \theta + \sin \theta \left[\cos \theta + \frac{2}{3} \cos^3 \theta + \dots \right. \right. \\ \left. \left. + \frac{2 \cdot 4 \cdot \dots \cdot (\nu-3)}{1 \cdot 3 \cdot \dots \cdot (\nu-2)} \cos^{\nu-2} \theta \right] \right\} & (\nu > 1 \text{ and odd}) \\ \frac{2}{\pi} \theta & (\nu = 1) \end{cases}$$

26.7.4

$$A(t|\nu) = \sin \theta \left\{ 1 + \frac{1}{2} \cos^2 \theta + \frac{1 \cdot 3}{2 \cdot 4} \cos^4 \theta + \dots \right. \\ \left. + \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (\nu-3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (\nu-2)} \cos^{\nu-2} \theta \right\} \quad (\nu \text{ even}) *$$

*See page 11.

Asymptotic Expansion for the Inverse Function

If $A(t_p|v) = 1 - 2p$ and $Q(x_p) = p$, then

26.7.5

$$t_p \sim x_p + \frac{g_1(x_p)}{v} + \frac{g_2(x_p)}{v^2} + \frac{g_3(x_p)}{v^3} + \dots$$

$$g_1(x) = \frac{1}{4} (x^3 + x)$$

$$g_2(x) = \frac{1}{96} (5x^5 + 16x^3 + 3x)$$

$$g_3(x) = \frac{1}{384} (3x^7 + 19x^5 + 17x^3 - 15x)$$

$$g_4(x) = \frac{1}{92160} (79x^9 + 776x^7 + 1482x^5 - 1920x^3 - 945x)$$

Limiting Distribution

26.7.6

$$\lim_{v \rightarrow \infty} A(t|v) = \frac{1}{\sqrt{2\pi}} \int_{-t}^t e^{-x^2/2} dx = A(t)$$

Approximation for Large Values of t and $v \leq 5$

$$26.7.7 \quad A(t|v) \approx 1 - 2 \left\{ \frac{a_v}{t^v} + \frac{b_v}{t^{v+1}} \right\}$$

v	1	2	3	4	5
a_v	.3183	.4991	1.1094	3.0941	9.948
b_v	.0000	.0518	-.0460	-2.756	-14.05

Approximation for Large v

$$26.7.8 \quad A(t|v) \approx 2P(x) - 1, \quad x = \frac{t \left(1 - \frac{1}{4v}\right)}{\sqrt{1 + \frac{t^2}{2v}}}$$

Non-Central t -Distribution

26.7.9

$$P(t'|v, \delta) =$$

$$\frac{1}{\sqrt{v} B\left(\frac{1}{2}, \frac{v}{2}\right)} \int_{-\infty}^{t'} \left(\frac{v}{v+x^2}\right)^{\frac{v+1}{2}} e^{-\frac{1}{2} \frac{x^2}{1+x^2}} H_h \left(\frac{-\delta x}{\sqrt{v+x^2}}\right) dx$$

$$= 1 - \sum_{j=0}^{\infty} e^{-\delta^2/2} \frac{(\delta^2/2)^j}{2j!} I_z\left(\frac{v}{2}, \frac{1}{2} + j\right), \quad x = \frac{v}{v+t'^2} *$$

where δ is termed the non-centrality parameter.

Approximation to the Non-Central t -Distribution

26.7.10

$$P(t'|v, \delta) \approx P(x) \quad \text{where } x = \frac{t' \left(1 - \frac{1}{4v}\right) - \delta}{\left(1 + \frac{t'^2}{2v}\right)^{1/2}}$$

Numerical Methods**26.8. Methods of Generating Random Numbers and Their Applications⁹**

Random digits are digits generated by repeated independent drawings from the population 0, 1, 2, . . . , 9 where the probability of selecting any digit is one-tenth. This is equivalent to putting 10 balls, numbered from 0 to 9, into an urn and drawing one ball at a time, replacing the ball after each drawing. The recorded set of numbers forms a collection of random digits. Any group of n successive random digits is known as a *random number*.

Several lengthy tables of random digits are available (see references). However, the use of random numbers in electronic computers has resulted in a need for random numbers to be generated in a completely deterministic way. The numbers so generated are termed pseudo-random numbers. The quality of pseudo-random numbers is determined by subjecting the numbers to several statistical tests, see [26.55], [26.56]. The purpose of these statistical tests is to detect any properties of the pseudo-random numbers which are different from the (conceptual) properties of random numbers.

⁹ The authors wish to express their appreciation to Professor J. W. Tukey who made many penetrating and helpful suggestions in this section.

Experience has shown that the congruence method is the most preferable device for generating random numbers on a computer. Let the sequence of pseudo-random numbers be denoted by $\{X_n\}$, $n=0, 1, 2, \dots$. Then the congruence method of generating pseudo-random numbers is

$$X_{n+1} = aX_n + b \pmod{T}$$

where b and T are relatively prime. The choice of T is determined by the capacity and base of the computer; a and b are chosen so that: (1) the resulting sequence $\{X_n\}$ possesses the desired statistical properties of random numbers, (2) the period of the sequence is as long as possible, and (3) the speed of generation is fast. A guide for choosing a and b is to make the correlation between the numbers be near zero, e.g., the correlation between X_n and X_{n+s} is

$$\rho_s = \frac{1 - 6 \frac{b_s}{T} \left(1 - \frac{b_s}{T}\right)}{a_s} + e$$

where

$$a_s = a^s \pmod{T}$$

$$b_s = (1 + a + a^2 + \dots + a^{s-1})b \pmod{T}$$

$$|e| < a_s/T$$

which occur in

$$X_{n+1} = a_n X_n + b_n \pmod{T}$$

When a is chosen so that $a \approx T^{1/2}$, the correlation $\rho_1 \approx T^{-1/2}$

The sequence defined by the multiplicative congruence method will have a full period of T numbers if

- (i) b is relatively prime to T
- (ii) $a \equiv 1 \pmod{p}$ if p is a prime factor of T
- (iii) $a \equiv 1 \pmod{4}$ if 4 is a factor of T .

Consequently if $T=2^s$, b need only be odd, and

$a \equiv 1 \pmod{4}$. When $T=10^s$, b need only be not divisible by 2 or 5, and $a \equiv 1 \pmod{20}$. The most convenient choices for a are of the form $a=2^s+1$ (for binary computers) and $a=10^s+1$ (for decimal computers). This results in the fastest generation of random numbers as the operations only require a shift operation plus two additions. Also any number can serve as the starting point to generate a sequence of random digits. A good summary of generating pseudo-random numbers is [26.51].

Below are listed various congruence schemes and their properties.

Congruence methods for generating random numbers

$$X_{n+1} = aX_n + b \pmod{T}, \quad T \text{ and } b \text{ relatively prime}$$

	a	b	T	Period	X_0	Special cases for which random numbers have passed statistical tests for randomness ¹²
26.8.1	$1+b$	odd	$T=t^s$	t^s	$0 \leq X_0 < T$	$T=2^{24}$, X_0 unknown; $a=2^2+1$, $b=1$; $T=2^{27}$, $a=2^2+1$, $b=29741\ 09825\ 8473$, $X_0=76293\ 94531\ 25$.
26.8.2	$2^s \pm 1$ (r odd, $s \geq 2$)	0	$T=t^s$	t^{s-1}	relatively prime to T	$T=2^{24}$, 2^s , $X_0=1$; $a=5^{11}(s=2)$ $T=2^{28}$, $X_0=1$; $T=2^{28}$, $X_0=1-2^{-28}$, $.5478126193$; $a=5^{11}(s=2)$ $T=2^{28}$, $X_0=1$; $a=5^{11}(s=2)$
26.8.3	$2^s \pm 1$ (r odd, $s \geq 2$)	0	$T=t^s \pm 1$	(varies)	relatively prime to T	$T=2^{24}+1$, $X_0=10,987,654,321$; $a=23$; period $\approx 10^8$ $T=10^8+1$, $X_0=47,594,118$; $a=23$; period $\approx 5.8 \times 10^8$
26.8.4	$7b+1$	0	$T=10^s$	$5 \cdot 10^{s-1}$	relatively prime to T	$T=10^{11}$, $X_0=1$; $a=7$ $T=10^{11}$, $X_0=1$; $a=7^{11}$
26.8.5	$3b+1$ ($s=0, 2, 3, 4$)	0	$T=10^s$	$5 \cdot 10^{s-1}$	relatively prime to T	

¹² X_0 given is the starting point for random numbers when statistical tests were made.

When the numbers are generated using a congruence scheme, the least significant digits have short periods. Hence the entire word length cannot be used. If one desired random numbers with as many digits as possible, one would have to modify the congruence schemes. One way is to generate the numbers mod $T \pm 1$. This unfortunately reduces the period.

Generation of Random Deviates

Let $\{X\}$ be a generated sequence of independent random numbers having the domain $(0, T)$. Then $\{U\} = \{T^{-1}X\}$ is a sequence of random deviates (numbers) from a uniform distribution on the interval $(0, 1)$. This is usually a necessary preliminary step in the generation of random deviates having a given cumulative distribution function $F(y)$ or probability density function $f(y)$. Below are summarized some general techniques

for producing arbitrary random deviates. (In what follows $\{U\}$ will always denote a sequence of random deviates from a uniform distribution on the interval $(0, 1)$.)

1. Inverse Method

The solutions $\{y\}$ of the equations $\{u = F(y)\}$ form a sequence of independent random deviates with cumulative distribution function $F(y)$. (If $F(y)$ has a discontinuity at $y=y_0$, then whenever u is such that $F(y_0-0) < u < F(y_0)$, select y_0 as the corresponding deviate.) Generally the inverse method is not practical unless the inverse function $y = F^{-1}(u)$ can be obtained explicitly or can be conveniently approximated.

2. Generating a Discrete Random Variable

Let Y be a discrete random variable with point probabilities $p_i = \text{Pr}\{Y=y_i\}$ for $i=1, 2, \dots$

The direct way to generate Y is to generate $\{U\}$ and put $Y=y_i$ if

$$p_1 + p_2 + \dots + p_{i-1} < U < p_1 + p_2 + \dots + p_i.$$

However, this method requires complicated machine programs that take too long.

An alternative way due to Marsaglia [26.53] is simple, fast, and seems to be well suited to high-speed computations. Let p_i for $i=1, 2, \dots, n$ be expressed by k decimal digits as $p_i = .\delta_{i1}\delta_{i2}\dots\delta_{ik}$ where the δ 's are the decimal digits. (If the domain of the random variable is infinite, it is necessary to truncate the probability distribution at p_n .) Define

$$P_0=0, P_r=10^{-r} \sum_{i=1}^n \delta_{ri} \text{ for } r=1, 2, \dots, k, \text{ and}$$

$$\Pi_s = \sum_{r=0}^k 10^r P_r, s=1, 2, \dots, k.$$

Number the computer memory locations by 0, 1, 2, \dots , Π_k-1 . The memory locations are divided into k mutually exclusive sets such that the s th set consists of memory locations $\Pi_{s-1}, \Pi_{s-1}+1, \dots, \Pi_s-1$. The information stored in the memory locations of the s th set consists of y_1 in δ_{s1} locations, y_2 in δ_{s2} locations, \dots , y_n in δ_{sn} locations.

Denote the decimal expansion of the uniform deviates generated by the computer by $u = .d_1d_2d_3\dots$ and finally let $\sigma\{m\}$ be the contents of memory location m . Then if

$$\sum_{i=0}^{s-1} P_i \leq U < \sum_{i=0}^s P_i$$

put

$$y = a \left\{ d_1d_2 \dots d_s + \Pi_{s-1} - 10^s \sum_{i=1}^{s-1} P_i \right\}.$$

This method is perhaps the best all-around method for generating random deviates from a discrete distribution. In order to illustrate this method consider the problem of generating deviates from the binomial distribution with point probabilities

$$p_i = \binom{n}{i} p^i (1-p)^{n-i}$$

for $n=5$ and $p=.20$. The point probabilities to 4 D are

Value of Random Variable	Point Probabilities
0	$p_0=0.3277$
1	$p_1=.4096$
2	$p_2=.2048$
3	$p_3=.0512$
4	$p_4=.0064$
5	$p_5=.0003$

and thus $P_0=0, P_1=.9, P_2=.07, P_3=.027, P_4=.0030$ from which $\Pi_0=0, \Pi_1=9, \Pi_2=16, \Pi_3=43, \Pi_4=73$. The 73 memory locations are divided into 4 mutually exclusive sets such that

Set	Memory Locations
1	0, 1, \dots , 8
2	9, 10, \dots , 15
3	16, \dots , 42
4	43, \dots , 72

Among the nine memory locations of set 1, zero is stored $\delta_{10}=3$ times, 1 is stored $\delta_{11}=4$ times, 2 is stored $\delta_{12}=2$ times; the seven locations of set 2 store 0 $\delta_{20}=2$ times and 3 $\delta_{23}=5$ times; etc. A summary of the memory locations is set out below:

	Value of Random Variable					
	0	1	2	3	4	5
Frequency (set 1)	3	4	2	0	0	0
Frequency (set 2)	2	0	0	5	0	0
Frequency (set 3)	7	9	4	1	6	0
Frequency (set 4)	7	6	8	2	4	3

Then to generate the random variables if

$0 \leq u < .9$	put	$y = a\{d_1\}$
$.9 \leq u < .97$		$y = a\{d_1d_2 - 81\}$
$.97 \leq u < .997$		$y = a\{d_1d_2d_3 - 954\}$
$.997 \leq u < 1.000$		$y = a\{d_1d_2d_3d_4 - 9927\}$

3. Generating a Continuous Random Variable

The method for generating deviates from a discrete distribution can be adapted to random variables having a continuous distribution. Let $F(y)$ be the cumulative distribution function and assume that the domain of the random variable is (a, b) where the interval is finite. (If the domain is infinite, it must be truncated at (say) the points a and b .) Divide the interval $(b-a)$ into n sub-intervals of length Δ ($n\Delta=b-a$) such that the boundary of the i th interval is (y_{i-1}, y_i) where $y_i=a+i\Delta$ for $i=0, 1, \dots, n$. Now define a discrete distribution having domain

$$\left\{ z_i = \frac{y_i + y_{i-1}}{2} \right\}$$

with point probabilities $p_i = F(y_i) - F(y_{i-1})$. Finally, let W be a random variable having a uniform distribution on $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$. This can be done by setting $W = \Delta\left(U - \frac{1}{2}\right)$. Then random

deviates from the distribution function $F(y)$, can be generated (approximately) by setting $y = z + w = z + \Delta \left(u - \frac{1}{2}\right)$. This is simply an approximate decomposition of the continuous random variable into the sum of a discrete and continuous random variable. The discrete variable can be generated quickly by the method described previously. The smaller the value of Δ the better will be the approximation. Each number can be generated by using the leading digits of U to generate the discrete random variable Z and the remaining digits forming a uniformly distributed deviate having $(0,1)$ domain.

4. Acceptance-Rejection Methods

In what follows the random variable Y will be assumed to have finite domain (a, b) . If the domain is infinite, it must be truncated for computational purposes at (say) the points a and b . Then the resulting random deviates will only have this truncated domain.

a) Let f be the maximum of $f(y)$. Then the procedure for generating random deviates is: (1) generate a pair of uniform deviates U_1, U_2 ; (2) compute a point $y = a + (b-a)u_2$ in (a, b) ; (3) if $u_1 < f(y)/f$ accept y as the random deviate, otherwise reject the pair (u_1, u_2) and start again. The acceptance ratio of deviates actually produced is $[(b-a)f]^{-1}$. Hence the acceptance ratio decreases as the domain increases. One way to increase the acceptance ratio is to divide the interval (a, b) into mutually exclusive sub-intervals and then carry out the acceptance-rejection process. For this purpose let the interval (a, b) be divided into k sub-intervals such that at the end points of the j th interval are (ξ_{j-1}, ξ_j) with $\xi_0 = a, \xi_k = b$ and $\int_{\xi_{j-1}}^{\xi_j} f(y) dy = p_j$; further let the maximum of $f(y)$ in the j th interval be f_j . Then to generate random deviates from $f(y)$, generate n pairs of deviates $(u_1, u_2), s = 1, 2, \dots, n$. Assign $[np_j]$ such pairs to the j th interval and compute $y_j = \xi_{j-1} + (\xi_j - \xi_{j-1})u_2$. If $u_1 < f(y_j)/f_j$ accept y_j as a deviate. The acceptance ratio of this method is

$$\sum_{j=1}^k p_j [(\xi_j - \xi_{j-1}) f_j]^{-1}$$

b) Let $F(y)$ be such that $f(y) = f_1(y)f_2(y)$ where the domain of y is (a, b) . Let f_1 and f_2 be the maximum of $f_1(y)$ and $f_2(y)$ respectively. Then the procedure for generating random de-

vates having the probability density function $f(y)$ is: (1) generate U_1, U_2, U_3 ; (2) define $z = a + (b-a)u_3$; (3) if both $u_1 < \frac{f_1(z)}{f_1}$ and $u_2 < \frac{f_2(z)}{f_2}$, take z as the random deviate; otherwise take another sample of three uniform deviates. The acceptance ratio of this method is $[(b-a)f_1f_2]^{-1}$ and can be increased by dividing (a, b) into sub-intervals as in the previous case.

c) Let the probability density function of Y be

$$f(y) = \int_a^{\beta} g(y, t) dt, (\alpha \leq t \leq \beta), (a \leq y \leq b).$$

Let g be the maximum of $g(y, t)$. Then the procedure for generating random deviates having the probability density function $f(y)$ is: (1) generate U_1, U_2, U_3 ; (2) define $s = \alpha + (\beta - \alpha)u_3$; $z = a + (b-a)u_3$; (3) if $u_1 < \frac{g(z, s)}{g}$, take z as the random deviate; otherwise take another sample of three. The acceptance ratio for this method is $[(b-a)g]^{-1}$ and can be increased by dividing the domain of t and y into sub-domains.

5. Composition Method

Let $g_z(y)$ be a probability density function which depends on the parameter z ; further let $H(z)$ be the cumulative distribution function for z . In order to generate random deviates Y having the frequency function

$$f(y) = \int_{-\infty}^{\infty} g_z(y) dH(z)$$

one draws a deviate having the cumulative distribution function $H(z)$; then draws a second sample having the probability density function $g_z(y)$.

6. Generation of Random Deviates From Well Known Distributions

a. Normal distribution

(1) *Inverse method*: The inverse method depends on having a convenient approximation to the inverse function $x = P^{-1}(u)$ where

$$u = (2\pi)^{-1/2} \int_{-\infty}^x e^{-t^2/2} dt.$$

Two ways of performing this operation are to (i) use 26.2.23 with $t = \left(\ln \frac{1}{u^2}\right)^{1/2}$ or (ii) approximate $x = P^{-1}(u)$ piecewise using Chebyshev polynomials, see [26.54].

(2) *Sum of uniform deviates*: Let U_1, U_2, \dots, U_n be a sequence of n uniform deviates. Then

$$X_n = \left(\sum_{i=1}^n U_i - \frac{n}{2} \right) \left(\frac{n}{12} \right)^{-1/2}$$

will be distributed asymptotically as a normal random deviate. When $n=12$, the maximum errors made in the normal deviate are 9×10^{-3} for $|X| < 2$, 9×10^{-1} for $2 < |X| < 3$. An improvement can be made by taking a polynomial function of X_n (say)

$$X_n^* = X_n \sum_{s=0}^k a_{2s} X_n^{2s}$$

as the normal deviate where a_{2s} are suitable coefficients. These coefficients may be calculated using (say) Chebyshev polynomials or simply by making the asymptotic random deviate agree with the correct normal deviate at certain specified points. When $n=12$, the maximum error in the normal deviate is 8×10^{-4} using the coefficients

$$\begin{array}{ll} * a_0 = 9.8746 & * a_6 = (-7) - 5.102 \\ * a_2 = (-3) 3.9439 & * a_8 = (-7) 1.141 \\ * a_4 = (-5) 7.474 & \end{array}$$

(3) *Direct method*: Generate a pair of uniform deviates (U_1, U_2) . Then

$$X_1 = (-2 \ln U_1)^{1/2} \cos 2\pi U_2,$$

$X_2 = (-2 \ln U_1)^{1/2} \sin 2\pi U_2$ will be a pair of independent normal random deviates with mean zero and unit variance. This procedure can be modified by calculating $\cos 2\pi U$ and $\sin 2\pi U$ using an acceptance rejection method; e.g., (1) generate (U_1, U_2) ; (2) if $(2U_1 - 1)^2 + (2U_2 - 1)^2 \leq 1$ generate a third uniform deviate U_3 , otherwise reject the pair and start over; (3) calculate $y_1 = (-\ln u_3)^{1/2} \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2}$, $y_2 = \pm 2(-\ln u_3)^{1/2} \frac{u_1 u_2}{u_1^2 + u_2^2}$ (\pm random). Both y_1 and y_2 are the desired random deviates.

(4) *Acceptance-rejection method*: 1) Generate a pair of uniform deviates (U_1, U_2) ; 2) compute $x = -\ln u_1$; 3) if $e^{-u(x-1)^2} \geq u_2$ (or equivalently $(x-1)^2 \leq -2(\ln u_2)$) accept x , otherwise reject the

pair and start over. The quantity will be the required normal deviate with mean zero and unit variance.

b. Bivariate normal distribution

Let $\{X_1, X_2\}$ be a pair of independent normal deviates with mean zero and unit variance. Then $\{X_1, \rho X_1 + (1 - \rho^2)^{1/2} X_2\}$ represent a pair of deviates from a bivariate normal distribution with zero means, unit variances, and correlation coefficient ρ .

c. Exponential distribution

(1) *Inverse method*: Since $F(x) = e^{-x/\theta}$, $X = -\theta \ln U$ will be a deviate from the exponential distribution with parameter θ .

(2) *Acceptance-rejection method*: 1) Generate a pair of independent uniform deviates (U_0, U_1) ; 2) if $U_1 < U_0$ generate a third value U_2 ; 3) if $U_1 + U_2 < U_0$ generate a fourth value U_3 , etc.; 4) continue generating uniform deviates until an n is obtained such that $U_1 + U_2 + \dots + U_{n-1} < U_0 < U_1 + \dots + U_n$; 5) if n is even reject the procedure and start a fresh trial with a new value of U_0 , otherwise if n is odd take $X = \theta U_0$ as the desired deviate; 6) in general if t is the number of trials until an acceptable sequence is obtained $X = \theta(t + U_0)$. The random deviates produced in this way follow an exponential distribution with parameter θ . One can expect to generate approximately six uniform deviates for every exponential deviate.

(3) *Discrete Distribution Method*: Let Y and n be discrete random variables with point probabilities

$$* Pr\{Y=r\} = (e-1)e^{-(r+1)} \quad r=0, 1, 2, \dots$$

$$Pr\{n=s\} = [s!(e-1)]^{-1} \quad s=1, 2, 3, \dots$$

Then $X = Y + \min(U_1, U_2, \dots, U_n)$ will follow an exponential distribution. The average value of n is 1.58 so that one needs, on the average, only 1.58 u 's from which the minimum is selected.

26.9. Use and Extension of the Tables

Use of Probability Function Inequalities

Example 1. Let X be a random variable with finite mean and variance equal to m and σ^2 , respectively. Use the inequalities for probability functions 26.1.37, 40, 41 to place lower bounds on

$$A(t) = F(t) - F(-t) = P\left\{ \frac{|X-m|}{\sigma} \leq t \right\}$$

for $t=1(1)4$.

Lower bounds on $A(t) = F(t) - F(-t)$

$t=1$	2	3	4	Remarks
0	.7500	.8889	.9375	no knowledge of $F(t)$; 26.1.37
.5556	.8889	.9506	.9722	$F(t)$ is unimodal and continuous; 26.1.40
0	.8182	.9697	.9912	$F(t)$ is such that $\mu_4=3$; 26.1.41

It is of interest to note that the standard normal distribution is unimodal, has mean zero, unit variance $\mu_4=3$, is continuous, and such that

$$A(t) = P(t) - P(-t) \\ = .6827, .9545, .9973, \text{ and } .9999$$

for $t=1, 2, 3$ and 4 respectively.

Interpolation for $P(x)$ in Table 26.1

Example 2. Compute $P(x)$ for $x=2.576$ to fifteen decimal places using a Taylor expansion.

Writing $x=x_0+\theta$ we have

$$P(x) = P(x_0) + Z(x_0)\theta + Z^{(1)}(x_0)\frac{\theta^2}{2!} \\ + Z^{(2)}(x_0)\frac{\theta^3}{3!} + Z^{(3)}(x_0)\frac{\theta^4}{4!} + \dots$$

Taking $x_0=2.58$ and $\theta=-4\times 10^{-3}$ we calculate the successive terms to 16D

+.99505	99842	42230	
—	5	72204	35976 6
—		2952	57449 6
—		8	63097 8
—			1439 4
—			9
<hr/>			
.99500	24676	84265	7

The result correct to 17D is

$$P(2.576) = .99500 \quad 24676 \quad 84264 \quad 98$$

Calculation for Arbitrary Mean and Variance

Example 3. Find the value to 5D of

$$P\{X \leq .50\} = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{.5} e^{-1/2\left(\frac{t-1}{2}\right)^2} dt$$

using 26.2.8 and Table 26.1.

This represents the probability of the random variable being less than or equal to .5 for a normal distribution with mean $m=1$ and variance $\sigma^2=4$. Using 26.2.8 we have

$$P\{X \leq .5\} = P\left(\frac{.5-1}{2}\right) = P(-.25)$$

Since $P(-x) = 1 - P(x)$, we have

$$P(-.25) = 1 - P(.25) = 1 - .59871 = .40129$$

where a two-term Taylor series was used for interpolation. Note that when interpolating for $P(x)$ for a value of x midway between the tabulated

values we can write $x=x_0+.01$ and a two-term Taylor series is $P(x)=P(x_0)+Z(x_0)10^{-2}$. Thus one need only multiply $Z(x_0)$ by 10^{-2} and add the result to $P(x_0)$.

Calculation of $P(x)$ for x Approximate

Example 4. Using Table 26.1, find $P(x)$ for $x=1.96$, when there is a possible error in x of $\pm 5\times 10^{-3}$.

This is an example where the argument is only known approximately. The question arises as to how many decimal places one should retain in $P(x)$. If Δx and $\Delta P(x)$ denote the error in x and the resulting error in $P(x)$, respectively, then

$$\Delta P(x) \approx Z(x)\Delta x$$

Hence $\Delta P(1.960) = 3\times 10^{-4}$ which indicates that $P(1.960)$ need only be calculated to 4D. Therefore $P(1.960) = .9750$.

Inverse Interpolation for $P(x)$

Example 5. Find the value of x for which $P(x) = .97500 \ 00000 \ 00000$ using Table 26.1 and determining as many decimal places as is consistent with the tabulated function.

For inverse interpolation the tabulated function $P(x)$ may be regarded as having a possible error of $.5\times 10^{-15}$. Hence

$$\Delta x \approx \frac{\Delta P(x)}{Z(x)} = \frac{.5\times 10^{-15}}{Z(x)}$$

Let $P(x_0)$ correspond to the closest tabulated value of $P(x)$. Then a convenient formula for inverse interpolation is

$$x = x_0 + t + \frac{x_0 t^2}{2} + \frac{2x_0^2 + 1}{6} t^3$$

where

$$t = \frac{P(x) - P(x_0)}{Z(x_0)}$$

If only the first two terms (i.e., $x=x_0+t$) are used, the error in x will be bounded by $\frac{x}{8}\times 10^{-4}$ and the true value will always be greater than the value thus calculated.

With respect to this example, $\Delta x \approx 10^{-14}$, and thus the interpolated value of x may be in error by one unit in the fourteenth place. The closest value to $P(x) = .97500 \ 00000 \ 00000$ is $P(x_0) = .97500 \ 21048 \ 51780$ with $x_0=1.96$. Hence using the preceding inverse interpolation formulas with

$$t = -.00003 \ 60167 \ 31129$$

and carrying fifteen decimals we have the successive terms

$$\begin{array}{r} +1.96000 \quad 00000 \quad 00000 \\ - \quad .00003 \quad 60167 \quad 31129 \\ + \quad \quad \quad 12 \quad 71261 \\ - \quad \quad \quad \quad \quad 68 \\ \hline +1.95996 \quad 39845 \quad 40064 \end{array}$$

Edgeworth Asymptotic Expansion

Example 6. Find the Edgeworth asymptotic expansion 26.2.49 for the c.d.f. of chi-square.

Method 1. Expansion for χ^2

Let

$$Q(\chi^2|\nu) = 1 - F(t)$$

where

$$t = \frac{\chi^2 - \nu}{(2\nu)^{1/2}}$$

Since the values of γ_1 and γ_2 26.4.33 are

$$\gamma_1 = 2\sqrt{2/\nu}$$

$$\gamma_2 = 12/\nu,$$

we obtain, by using the first two bracketed terms of 26.2.49

$$F(t) \sim P(t) - \frac{1}{\nu^{1/2}} \left[\frac{\sqrt{2}}{3} Z^{(2)}(t) \right] + \frac{1}{\nu} \left[\frac{1}{2} Z^{(3)}(t) + \frac{1}{9} Z^{(5)}(t) \right]$$

The Edgeworth expansion is an asymptotic expansion in terms of derivatives of the normal distribution function. It is often possible to transform a random variable so that the distribution of the transformed random variable more closely approximates the normal distribution function than does the distribution of the original random variable. Hence for the same number of terms, greater accuracy may be achieved by using the transformed variable in the expansion. Since the distribution of $\sqrt{2\chi^2}$ is more closely approximated by a normal distribution than χ^2 itself (as judged by a comparison of the values of γ_1 and γ_2), we would expect that the Edgeworth asymptotic expansion of $\sqrt{2\chi^2}$ would be superior to that of χ^2 .

Method 2. Expansion for $\sqrt{2\chi^2}$. Let

$$Q(\chi^2|\nu) = 1 - F(t) = 1 - F\left(\frac{\sqrt{2\chi^2} - (2\nu-1)^{1/2}}{(1-\frac{1}{4\nu})^{1/2}}\right)$$

where $(2\nu-1)^{1/2}$ and $1-\frac{1}{4\nu}$ are the mean and variance to terms of order ν^{-2} of $\sqrt{2\chi^2}$ (see 26.4.34). The values of γ_1 and γ_2 for $\sqrt{2\chi^2}$ are

$$\gamma_1 \approx \frac{1}{\sqrt{2\nu}} \left[1 + \frac{5}{8\nu} \right] \quad \gamma_2 \approx \frac{3}{4\nu^2}$$

Thus we obtain

$$F(t) \sim P(t) - \frac{1}{\nu^{1/2}} \left[\frac{\sqrt{2}}{12} \left(1 + \frac{5}{8\nu} \right) Z^{(2)}(t) \right] + \frac{1}{\nu} \left[\frac{1}{32\nu} Z^{(3)}(t) + \frac{1}{144} \left(1 + \frac{5}{8\nu} \right)^2 Z^{(5)}(t) \right]$$

For numerical examples using these expansions see **Example 12.**

Calculation of $L(h, k, \rho)$

Example 7. Find $L(.5, .4, .8)$. Using 26.3.20

$$\sqrt{h^2 - 2\rho hk + k^2} = \sqrt{.09} = .3$$

$$L(.5, .4, .8) = L(.5, 0, 0) + L(.4, 0, -.6)$$

Reference to **Figure 26.2** yields

$$L(.5, 0, 0) + L(.4, 0, -.6) = .16 + .08 = .24$$

The answer to 3D is $L(.5, .4, .8) = .250$.

Calculation of the Bivariate Normal Probability Function

Example 8. Let X and Y follow a bivariate normal distribution with parameters $m_x=3$, $m_y=2$, $\sigma_x=4$, $\sigma_y=2$, and $\rho=-.125$. Find the value of $P_r\{X \geq 2, Y \geq 4\}$ using 26.3.20 and **Figures 26.2, 26.3.**

Since $P_r\{X \geq h, Y \geq k\} = L\left(\frac{h-m_x}{\sigma_x}, \frac{k-m_y}{\sigma_y}, \rho\right)$ we have $P\{X \geq 2, Y \geq 4\} = L(-.25, 1, -.125)$. Using 26.3.20

$$L(-.25, 1, -.125) = L(-.25, 0, .969)$$

$$+ L(1, 0, .125) - \frac{1}{2}$$

Figure 26.2 only gives values for $h > 0$, however, using the relationship 26.3.8 with $k=0$, $L(-h, 0, \rho) = \frac{1}{2} - L(h, 0, -\rho)$ and thus $L(-.25, 0, .969) = \frac{1}{2} - L(.25, 0, -.969)$. Therefore $L(-.25, 1, -.125) = -L(.25, 0, -.969) + L(1, 0, .125) = -.01 + .09 = .08$. The answer to 3D is $L(-.25, 1, -.125) = .080$.

Integral of a Bivariate Normal Distribution Over a Polygon

Example 9. Let the random variables X and Y have a bivariate normal distribution with parameters $m_x=5$, $\sigma_x=2$, $m_y=9$, $\sigma_y=4$, and $\rho=.5$. Find the probability that the point (X, Y) be inside the triangle whose vertices are $A=(7, 8)$, $B=(9, 13)$, and $C=(2, 9)$.

When obtaining the integral of a bivariate normal distribution over a polygon, it is first necessary to use 26.3.22 in order to transform the variates so that one deals with a circular normal distribution. The polygon in the region of the transformed variables is then divided into configurations such that the integral over any selected configuration can be easily obtained. Below are listed some of the most useful configurations.

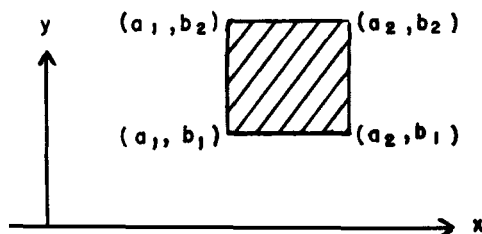


FIGURE 26.5

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} g(x, y, 0) dx dy = [P(a_2) - P(a_1)] [P(b_2) - P(b_1)]$$

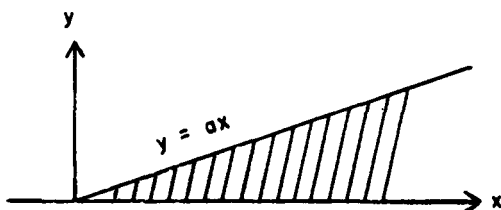


FIGURE 26.6

$$\int_0^\infty \int_0^{ax} g(x, y, 0) dx dy = \frac{\arctan a}{2\pi}$$

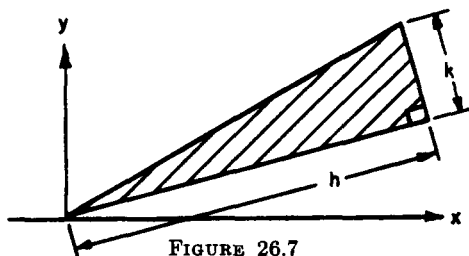


FIGURE 26.7

$$\int_0^h \int_0^{kx/h} g(x, y, 0) dx dy = V(h, k)^{11}$$

¹¹ See 26.3.23 for definition of $V(h, k)$.

For the following two configurations we define

$$h = \frac{|t_2 s_1 - t_1 s_2|}{[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{1/2}}$$

$$k_1 = \frac{|s_1(s_2 - s_1) + t_1(t_2 - t_1)|}{[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{1/2}}$$

$$k_2 = \frac{|s_2(s_2 - s_1) + t_2(t_2 - t_1)|}{[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{1/2}}$$

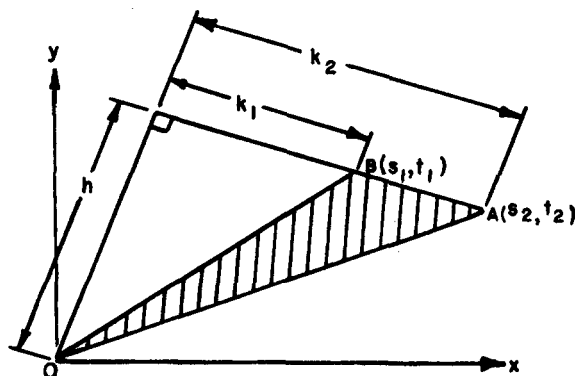


FIGURE 26.8

$$\iint_{\Delta AOB} g(x, y, 0) dx dy = V(h, k_2) - V(h, k_1)$$

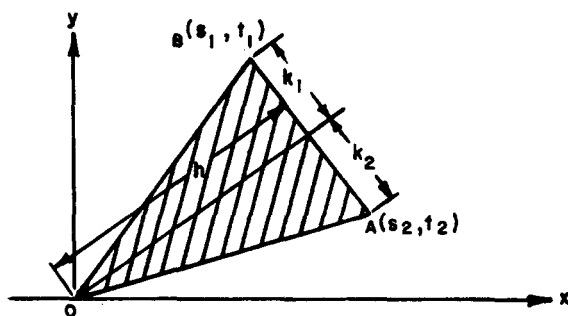


FIGURE 26.9

$$\iint_{\Delta AOB} g(x, y, 0) dx dy = V(h, k_2) + V(h, k_1)$$

Using the circularizing transformation 26.3.22 for our example results in

$$s = \frac{1}{\sqrt{3}} \left(\frac{x-5}{2} + \frac{y-9}{4} \right)$$

$$t = -\frac{1}{1} \left(\frac{x-5}{2} - \frac{y-9}{4} \right)$$

The vertices of the triangle in the (s, t) coordinates become $A = (\sqrt{3}/4, -5/4)$, $B = (\sqrt{3}, -1)$ and $C = (-\sqrt{3}/2, 3/2)$. These points are plotted below. From the figure it is seen that the desired probability is the sum of the probabilities that the point having the transformed variables as coordinates is inside the triangles AOB , AOC , and BOC .

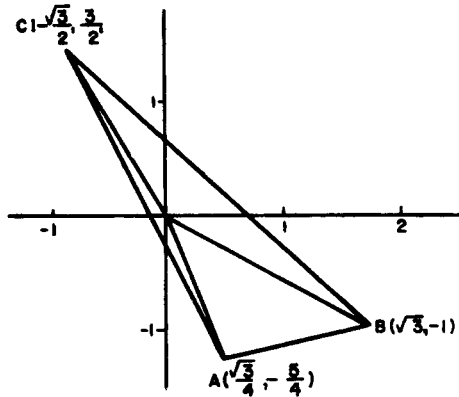


FIGURE 26.10

For these three triangles we have

	h	k_1	k_2
ΔAOB	$\frac{2}{7}\sqrt{21}$	$\sqrt{7}/14$	$\frac{4}{7}\sqrt{7}$
ΔAOC	$\frac{1}{74}\sqrt{111}$	$\frac{8}{37}\sqrt{37}$	$\frac{21}{74}\sqrt{37}$
ΔBOC	$\frac{1}{13}\sqrt{39}$	$\frac{7}{13}\sqrt{13}$	$\frac{6}{13}\sqrt{13}$

From the graph it is seen that the probability over AOB may be found in the same manner as that over Figure 26.8, and over AOC and BOC the probabilities may be found as that over Figure 26.9.

Hence

$$\begin{aligned} \iint_{\Delta} g(x, y, .5) dx dy &= \iint_{\Delta ABC} g(s, t, 0) ds dt \\ &= \iint_{\Delta AOB} g(s, t, 0) ds dt + \iint_{\Delta AOC} g(s, t, 0) ds dt \\ &\quad + \iint_{\Delta BOC} g(s, t, 0) ds dt \end{aligned}$$

and consequently using 26.3.23 and Figure 26.2

$$\begin{aligned} \iint_{\Delta AOB} g(s, t, 0) ds dt &= V\left(\frac{2}{7}\sqrt{21}, \frac{4\sqrt{7}}{7}\right) - V\left(\frac{2}{7}\sqrt{21}, \frac{\sqrt{7}}{14}\right) \\ &= \left[\frac{1}{4} + L(1.31, 0, -.76) - L(0, 0, -.76) - \frac{1}{2} Q(1.31)\right] \\ &\quad - \left[\frac{1}{4} + L(1.31, 0, -.14) - L(0, 0, -.14) - \frac{1}{2} Q(1.31)\right] \\ &= L(1.31, 0, -.76) - L(0, 0, -.76) \\ &\quad - L(1.31, 0, -.14) + L(0, 0, -.14) \\ &= .00 - .11 - .04 + .23 = .08 \end{aligned}$$

$$\begin{aligned} \iint_{\Delta AOC} g(s, t, 0) ds dt &= V\left(\frac{\sqrt{111}}{74}, \frac{8\sqrt{37}}{37}\right) + V\left(\frac{\sqrt{111}}{74}, \frac{21\sqrt{37}}{74}\right) \\ &= \left[\frac{1}{4} + L(.14, 0, -.99) - L(0, 0, -.99) - \frac{1}{2} Q(.14)\right] \\ &\quad + \left[\frac{1}{4} + L(.14, 0, -1) - L(0, 0, -1) - \frac{1}{2} Q(.14)\right] \\ &= .01 + .02 = .03 \end{aligned}$$

$$\begin{aligned} \iint_{\Delta BOC} g(s, t, 0) ds dt &= V\left(\frac{\sqrt{39}}{13}, \frac{7\sqrt{13}}{13}\right) + V\left(\frac{\sqrt{39}}{13}, \frac{6\sqrt{13}}{13}\right) \\ &= \left[\frac{1}{4} + L(.48, 0, -.97) - L(0, 0, -.97) - \frac{1}{2} Q(.48)\right] \\ &\quad + \left[\frac{1}{4} + L(.48, 0, -.96) - L(0, 0, -.96) - \frac{1}{2} Q(.48)\right] \\ &= .05 + .04 = .09 \end{aligned}$$

Thus adding all parts, the probability that X and Y are in triangle ABC is $.08 + .03 + .09 = .20$. The answer to 3D is .211.

Calculation of a Circular Normal Distribution Over an Offset Circle

Example 10. Let X and Y have a circular normal distribution with $\sigma = 1000$. Find the probability that the point (X, Y) falls within a circle having a radius equal to 540 whose center is displaced 1210 from the mean of the circular normal distribution.

In units of σ , the radius and displacement from the center are, respectively, $R = \frac{540}{1000} = .54$ and $r = \frac{1210}{1000} = 1.21$. The problem is thus reduced to finding the probability of X and Y falling in a circle of radius $R = .54$ displaced $r = 1.21$ from the center of the distribution where $\sigma = 1$.

Since $R < 1$, the approximation 26.3.25 is used. This results in

$$P(R^2|2, r^2) = \frac{2(.54)^2}{4 + (.54)^2} \exp \frac{-2(1.21)^2}{4 + (.54)^2} \\ = (.1359)e^{-.6823} = .06869$$

The answer to 5D is .06870.

Interpolation for $Q(x^2|\nu)$

Example 11. Find $Q(25.298|20)$ using the interpolation formula given with Table 26.7.

Taking $x^2 = 25$, $\theta = .298$ and applying the interpolation formula results in

$$Q(25.298|20) = \frac{1}{8} \{ Q(25|16)\theta^2 + Q(25|18)(4\theta - 2\theta^2) \\ + Q(25|20)(8 - 4\theta + \theta^2) \} \\ = \frac{1}{8} \{ (.06982)(.088804) \\ + (.12492)(1.014392) \\ + (.20143)(6.896804) \} \\ = .19027$$

A less accurate interpolate may be obtained by setting θ^2 equal to zero in the above formula. This results in the value .19003. The correct value to 6D is $Q(25.298|20) = .190259$.

On the other hand if $x^2 = 25.298$ is assumed to have an error of $\pm 5 \times 10^{-4}$, then how large an error arises in $Q(x^2|\nu)$? Denoting the error in x^2 by Δx^2 and the resulting error in $Q(x^2|\nu)$ by $\Delta Q(x^2|\nu)$, we then have the approximate relationship

$$\Delta Q(x^2|\nu) \approx \frac{\partial Q(x^2|\nu)}{\partial x^2} \Delta x^2$$

Using 26.4.8 we can write

$$\frac{\partial Q(x^2|\nu)}{\partial x^2} = \frac{1}{2} [Q(x^2|\nu-2) - Q(x^2|\nu)]$$

and

$$\Delta Q(x^2|\nu) \approx \frac{1}{2} [Q(x^2|\nu-2) - Q(x^2|\nu)] \Delta x^2$$

For practical purposes it is sufficient to evaluate the derivative to one or two significant figures. Consequently we can write

$$\frac{\partial Q(x^2|\nu)}{\partial x^2} \approx \frac{\partial Q(x_0^2|\nu)}{\partial x^2}$$

where x_0^2 is the closest value to x^2 for which Q is tabulated. Hence

$$\Delta Q(x^2|\nu) \approx \frac{1}{2} [Q(x_0^2|\nu-2) - Q(x_0^2|\nu)] \Delta x^2$$

For this example $\Delta x^2 = \pm 5 \times 10^{-4}$ and $x_0^2 = 25$. This results in

$$\Delta Q(x^2|\nu) = \frac{1}{2} (-.076)(\pm 5)10^{-4} = \pm 2 \times 10^{-5}$$

as the possible error in $Q(x^2|\nu)$.

Calculation of $Q(x^2|\nu)$ Outside the Range of Table 26.7

Example 12. Find the value of $Q(84|72)$.

Since this value is outside the range of Table 26.7 we can approximate $Q(84|72)$ by (1) using the Edgeworth expansion for $Q(x^2|\nu)$ given in Example 6, (2) the cube root approximation 26.4.14, (3) the improved cube root approximation 26.4.15 or (4) the square root approximation 26.4.13. The results of using all four methods are presented below:

1. Edgeworth expansion

The successive terms of the Edgeworth expansion for the distribution of chi-square result in

$$1 - Q(84|72) = .841345 \\ .000000 \\ .001120 \\ \hline .842465$$

Hence $Q(84|72) = .15754$.

The successive terms of the Edgeworth expansion for the distribution of $\sqrt{2x^2}$ result in

$$1 - Q(84|72) = .842544 \\ -.000034 \\ -.000138 \\ \hline .842372$$

Hence $Q(84|72) = .15764$.

2. Cube root approximation 26.4.14

Using the cube root approximation we have

$$Q(84|72) = Q(x)$$

where

$$x = \frac{\left(\frac{84}{72}\right)^{1/3} \left[1 - \frac{2}{9(72)}\right]}{\left[\frac{2}{9(72)}\right]^{1/3}} = 1.0046$$

This results in $Q(84|72) = Q(1.0046) = 1 - P(1.0046) = .15754$.

3. Improved cube root approximation 26.4.15

The improved cube root approximation involves calculating a correction factor h_r to x . Linearly interpolating for h_{80} (which appears below 26.4.15) with $x = 1.0046$ results in $h_{80} = -.0006$ and hence

$$h_{72} = \frac{60}{72}(-.0006) = -.00049$$

Thus

$$Q(84|72) = Q(1.0046 - .0005) = Q(1.0041) \\ = 1 - P(1.0041) = .15766$$

4. Square root approximation 26.4.13

Using the square root approximation we have $Q(84|72) = Q(x)$ where

$$x = \sqrt{2(84)} - \sqrt{2(72) - 1} = 1.0032.$$

This results in

$$Q(84|72) = Q(1.0032) = 1 - P(1.0032) = .15788$$

The value correct to 6D is $Q(84|72) = .157653$. Generally the improved cube root approximation will be correct with a maximum error of a few units in the fifth decimal and is recommended for calculations which are outside the range of **Table 26.7**.

Calculation of χ^2 for $Q(\chi^2|\nu)$ Outside the Range of **Table 26.8**

Example 13. Find the value of χ^2 for which $Q(\chi^2|144) = .01$.

Since $\nu = 144$ is outside the range of **Table 26.8**, we can compute it by using (1) the Cornish-Fisher asymptotic expansion 26.2.50, for χ^2 , (2) the cube approximation 26.4.17, (3) the improved cube approximation 26.4.18, or (4) the square approximation 26.4.16. We shall compute the value by all four methods.

1. Cornish-Fisher asymptotic expansion 26.2.50

The Cornish-Fisher asymptotic expansion for χ^2 with $\nu = 144$ can be written as

$$\chi^2 \sim 144 + 12\sqrt{2}x + 4h_1(x) + \frac{4\sqrt{2}}{12}[3h_2(x) + 2h_{11}(x)] \\ + \frac{8}{12^2}[6h_3(x) + 3h_{12}(x) + 2h_{111}(x)] + \frac{16\sqrt{2}}{12^3}[30h_4(x) \\ + 9h_{22}(x) + 12h_{13}(x) + 6h_{112}(x) + 4h_{1111}(x)]$$

Hence using the auxiliary table following 26.2.51 with $p = .01$ we have

144. 0000
39. 4794
2. 9413
— . 0242
— . 0019
+ . 0002
<hr/>
$\chi^2=186.395$

2. Cube approximation 26.4.17

Taking $\chi_{.01} = 2.32635$ we have

$$\chi^2 = 144 \left\{ \left[1 - \frac{2}{9(144)} \right] + (2.32635) \sqrt{\frac{2}{9(144)}} \right\}^3 = 186.405$$

3. Improved cube approximation 26.4.18

From the table for h_{60} we obtain using linear interpolation with $x = 2.33$ (approximately)

$$h_{60} = .0012 \text{ and thus } h_{144} = \frac{60}{144}(.0012) = .00049$$

Hence

$$\chi^2 = 144 \left[1 - \frac{2}{9(144)} + (2.32635 - .00049) \sqrt{\frac{2}{9(144)}} \right]^3 = 186.394$$

4. Square approximation 26.4.16

$$\chi^2 = \frac{1}{2} [2.32635 + \sqrt{2(144) - 1}]^2 = 185.616$$

The correct answer to 3D is $\chi^2 = 186.394$. Generally the improved cube approximation will give results correct in the second or third decimal for $\nu > 30$.

Calculation of the Incomplete Gamma Function

Example 14. Find the value of

$$\gamma(2.5, .9) = \int_0^{.9} t^{1.5} e^{-t} dt$$

making use of 26.4.19 and **Table 26.7**.

Using 26.4.19 we have

$$\gamma(2.5, .9) = \Gamma(2.5)P(1.8|5) = \Gamma(2.5)[1 - Q(1.8|5)]$$

$$\gamma(2.5, .9) = \frac{3}{4} \sqrt{\pi} [1 - .87607] = .16475$$

Poisson Distribution

Example 15. Find the value of m for which

$$\sum_{i=0}^3 e^{-m} \frac{m^i}{i!} = .99$$

using 26.4.21 and **Table 26.8**.

From **Table 26.8** with $\nu = 2c = 8$ and $Q = .99$ we have $\chi^2 = 1.646482$. Hence $m = \chi^2/2 = .823241$.

Inverse of the Incomplete Beta Function

Example 16. Find the value of x for which $I_x(10, 6) = .10$ using **Table 26.9** and 26.5.28.* Using 26.5.28 we have

*See page II.

$$I_z(10, 6) = Q(F|12, 20) = .10 \text{ where } x = \frac{20}{20+12F}$$

From **Table 26.9** the upper 10 percent point of F with 12 and 20 degrees of freedom is $F=1.89$. Hence

$$x = \frac{20}{20+12(1.89)} = .469$$

The correct value to 4D is $x=.4683$.

Calculation of $I_z(a, b)$ for a or b Small Integers

Example 17. Calculate $I_{.10}(3, 20)$.

Values of $I_z(a, b)$ for small integral a or b can conveniently be calculated using **26.5.6** or **26.5.7**. Using **26.5.6** we have

$$1 - I_{.90}(20, 3) = \frac{(.9)^{20}}{B(3, 20)} \left\{ \sum_{i=0}^2 (-1)^i \binom{2}{i} \frac{.9^i}{20+i} \right\} \\ = \frac{.121576}{.216450 \times 10^{-3}} (.110390 \times 10^{-2}) = .620040$$

Binomial Distribution

Example 18. Find the value of p which satisfies

$$\sum_{s=0}^{20} \binom{50}{s} p^s q^{50-s} = .95, \quad q = 1 - p$$

using **26.5.24** and **Table 26.9**.

* Combining **26.5.24** and **26.5.28** we have

$$\sum_{s=a}^n \binom{n}{s} p^s q^{n-s} = Q(F|\nu_1, \nu_2)$$

where

$$\nu_1 = 2(n-a+1), \nu_2 = 2(a), \text{ and } p = \frac{a}{a+(n-a+1)F}$$

Hence

$$\sum_{s=0}^{20} \binom{50}{s} p^s q^{50-s} = 1 - \sum_{s=21}^{50} \binom{50}{s} p^s q^{50-s} \\ = 1 - Q(F|60, 42) = .95$$

Harmonic interpolation on ν_2 in the table for which $Q(F|\nu_1, \nu_2) = .05$ results in $F=1.624$ for $\nu_1=60$, $\nu_2=42$, and thus $p = \frac{42}{42+60(1.624)} = .301$. The correct answer to 4D is $p = .3003$.

Approximating the Incomplete Beta Function

Example 19. Find $I_{.90}(16, 10.5)$ using **26.5.21**.

Values of $I_z(a, b)$ can conveniently be calculated with good accuracy using the approximation given by **26.5.20** or **26.5.21**. For this example $(a+b-1)(1-x) = 10.20$ which is greater than .8 and hence **26.5.21** will be used. Thus

$$w_1 = [(10.5)(.60)]^{1/3} = 1.8469, w_2 = [16(.4)]^{1/3} = 1.8566$$

$$y = \frac{3[(1.8469)(.98942) - (1.8566)(.99306)]}{\left[\frac{(1.8469)^2}{10.5} + \frac{(1.8566)^2}{16} \right]} = -.0668$$

and interpolating in **Table 26.1** gives

$$P(-.0668) = 1 - P(.0668) = .47336$$

The answer correct to 5D is $I_{.90}(16, 10.5) = .47332$.

Interpolation for F in **Table 26.9**

Example 20. Find the value of F for which

$$Q(F|7, 20) = .05 \text{ using } \textbf{Table 26.9}.$$

Interpolation in **Table 26.9** is approximately linear when the reciprocals of the degrees of freedom (ν_1, ν_2) are used as the interpolating variable. For this example it is only necessary to interpolate with respect to $1/\nu_1$. Thus linear interpolation on $1/\nu_1$ results in $F=2.51$ which is the correct interpolate.

Calculation of F for $Q(F|\nu_1, \nu_2) > .50$

Example 21. Find the value of F for which $Q(F|4, 8) = .90$ using **26.6.9** and **Table 26.9**.

Table 26.9 only tabulates values of F for which $Q(F|\nu_1, \nu_2) = p$ where $p = .500, .250, .100, .050, .025, .010, .005, .001$. However making use of **Table 26.9** we can find the values of F_p for which $p = .75, .9, .95, .975, .99, .995, .999$. For this example we have

$$F_{.90}(4, 8) = \frac{1}{F_{.10}(8, 4)}$$

and referring to the table for which $Q(F|\nu_1, \nu_2) = .10$ gives $F_{.10}(8, 4) = 3.95$ and thus $F_{.90}(4, 8) = \frac{1}{3.95} = .253$.

Calculation of $Q(F|\nu_1, \nu_2)$ for Small Integral ν_1 or ν_2

Example 22. Compute $Q(2.5|4, 15)$ using **26.6.4**.

Values of $Q(F|\nu_1, \nu_2)$ can be readily computed for small ν_1 or ν_2 using the expansions **26.6.4** to **26.6.8** inclusive. We have using **26.6.4**

$$x = \frac{15}{15+4(2.50)} = .60$$

and

$$Q(2.50|4, 15) = (.6)^{7.5} \left[1 + \frac{15}{2} (.4) \right] = .086735$$

Approximating $Q(F|v_1, v_2)$

Example 23. Calculate $Q(1.714|10, 40)$ using 26.6.15.

The approximation given by 26.6.15 will result in a maximum error of .0005. For this example we have

$$x = \frac{(1.714)^{1/3} \left(1 - \frac{2}{9(40)}\right) - \left(1 - \frac{2}{9(10)}\right)}{\left[\frac{2}{9(10)} + (1.714)^{2/3} \frac{2}{9(40)}\right]^{1/2}} = 1.2222$$

Interpolating in Table 26.1 results in

$$Q(1.714|10, 40) \approx Q(1.2222) = 1 - P(1.2222) = .1108$$

The correct value to 5D is $Q(1.714|10, 40) = .11108$.

On the other hand the approximation given by 26.6.14 which is usually less accurate results in

$$x = \frac{\sqrt{[2(40)-1] \left(\frac{10}{40}\right) (1.714) - \sqrt{2(10)-1}}}{\sqrt{1 + \frac{10}{40} (1.714)}} = 1.2210$$

and interpolating in Table 26.1 gives

$$Q(1.714|10, 40) \approx Q(1.2210) = 1 - P(1.2210) = .1112$$

Calculation of F Outside the Range of Table 26.9

Example 24. Find the value of F for which $Q(F|10, 20) \approx .0001$ using 26.6.16 and 26.5.22.

For this problem we have $a = \frac{v_2}{2} = 10$, $b = \frac{v_1}{2} = 5$, $p = .0001$. The value of the normal deviate which cuts off .0001 in the tail of the distribution is

$y = 3.7190$ (i.e., $Q(3.7190) = .0001$). Hence substituting in 26.5.22 gives

$$h = 2 \left[\frac{1}{19} + \frac{1}{9} \right]^{-1} = 12.2143$$

$$\lambda = \frac{3.7190^2 - 3}{6} = 1.8052$$

$$w = 3.7190 \frac{(12.2143 + 1.8052)^{1/2}}{12.2143}$$

$$- \left(\frac{1}{9} - \frac{1}{19} \right) \left[1.8052 + .8333 - \frac{2}{3(12.2143)} \right]$$

$$w = .9889$$

and thus $F \approx e^{2w} = 7.23$. The correct answer is $F = 7.180$.

Approximating the Non-Central F -Distribution

Example 25. Compute $P(3.71|3, 10, 4)$ using the approximation 26.6.27 to the non-central F -distribution.

Using 26.6.27 with $v_1 = 3$, $v_2 = 10$, $\lambda = 4$, $F' = 3.71$ we have

$$x = \frac{\left[\left(\frac{3}{3+4} \right) (3.71) \right]^{1/3} \left[1 - \frac{2}{9(10)} \right] - \left[1 - \frac{2}{9} \frac{(3+8)}{(3+4)^2} \right]}{\left[\frac{2}{9} \frac{3+8}{(3+4)^2} + \frac{2}{9(10)} \left[\left(\frac{3}{3+4} \right) (3.71) \right]^{2/3} \right]^{1/2}} = .675$$

and interpolating in Table 26.1 gives

$$P(3.71|3, 10, 4) \approx P(.675) = .750$$

The exact answer is $P(3.71|3, 10, 4) = .745$.

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- [26.22] D. B. Owen, Tables for computing bivariate normal probabilities, Ann. Math. Statist. 27, 1075-1090 (1956). $T(h, a) = \frac{1}{2\pi} \arctan a - V(h, ah)$ for $a=.25(.25)1$, $h=0(.01)2(.02)3$; $a=0(.01)1$, ∞ , $h=0(.25)3$; $a=.1$, $.2(.05).5(.1)8$, 1 , ∞ , $h=3(.05)3.5(.1)4.7$, 6D.
- [26.23] D. B. Owen, The bivariate normal probability function, Office of Technical Services, U.S. Department of Commerce (1957). $T(h, a) = \frac{1}{2\pi} \arctan a - V(h, ah)$ for $a=0(.025)1$, ∞ ; $h=0(.01)3.5(.05)4.75$, 6D.
- [26.24] Tables VIII and IX, Part II of [26.12]. $L(h, k, \rho)$ for $h, k=0(.1)2.6$, $\rho=-1(.05)1$, 6D for $\rho > 0$ and 7D for $\rho < 0$.
- Chi-Square, Non-Central Chi-Square, Probability Integral, Incomplete Gamma Function, Poisson Distribution**
- [26.25] G. A. Campbell, Probability curves showing Poisson's exponential summation, Bell System Technical Journal, 95-113 (1923). Tabulates values of $m = \frac{\chi^2}{2}$ for which $Q(\chi^2/\nu) = .000001$, 2D; .0001, .01, 3D; .1, .25, .5, .75, .9, 4D; .99, .9999, 3D; .999999, 2D for $c = \frac{\nu}{2} = 1(1)101$.
- [26.26] Table IV of [26.7]. Tabulates values of χ^2 for $Q(\chi^2/\nu) = .001, .01, .02, .05, .1, .2, .3, .5, .7, .8, .9, .95, .98, .99$ and $\nu=1(1)30$, 3D or 3S.
- [26.27] E. Fix, Tables of noncentral χ^2 , Univ. of California Publications in Statistics 1, 15-19 (1949). Tabulates λ for $P(\chi^2/\nu, \lambda) = .1(.1).9$, $Q(\chi^2/\nu) = .01, .05$; $\nu=1(1)20(2)40(5)60(10)100$, 3D or 3S.
- [26.28] H. O. Hartley and E. S. Pearson, Tables of the χ^2 integral and of the cumulative Poisson distribution, Biometrika 37, 313-325 (1950). Also reproduced as Table 7 in [26.11]. $P(\chi^2/\nu)$ for $\nu=1(1)20(2)70$, $\chi^2=0(.001).01(.01).1(.1)2(.2)10(.5)20(1)40(2)134$, 5D.
- [26.29] T. Kitagawa, Tables of Poisson distribution (Baifukan, Tokyo, Japan, 1951). $e^{-m}m^s/s!$ for $m=.001(.001)1(.01)5$, 8D; $m=5(.01)10$, 7D.
- [26.30] E. C. Molina, Poisson's exponential binomial limit (D. Van Nostrand Co., Inc., New York, N.Y., 1940). $e^{-m}m^s/s!$ and $P(\chi^2/\nu) = \sum_{j=t}^{\infty} e^{-m}m^j/j!$ for $m=\chi^2/2=0(.1)16(1)100$, 6D; $m=0(.001).01(.01)3$, 7D.
- [26.31] K. Pearson (Editor), Tables of the incomplete Γ -function, Biometrika Office, University College (Cambridge Univ. Press, Cambridge, England, 1934). $I(u, p)$ for $p=-1(.05)0(.1)5(.2)50$, $u=0(.1)$ $I(u, p)=1$ to 7D; $p=-1(.01)-.75$, $u=0(.1)6$, 5D; $\ln[I(u, p)|u^{p+1}]$, $p=-1(.05)0(.1)10$, $u=0(.1)1.5$, 8D; $[x^{p+1}\Gamma(p+1)]^{-1}\gamma(p, x)$, $p=-1(.01)-.9$, $x=0(.01)3$, 7D.
- [26.32] E. E. Sluckii, Tablitsy dlya vychioleniya nepolnoy Γ -funktsii i funktsii veroyatnosti χ^2 . (Izdat. Akad. Nauk SSSR, Moscow-Leningrad, U.S.S.R., 1950). $\Gamma(\chi^2, \nu) = \left(\frac{1}{2} \chi^2\right)^{-\nu/2} P(\chi^2/\nu)$, $\mathcal{P}(t, \nu) = Q(\chi^2/\nu)$, $\Pi(t, x) = Q(\chi^2/\nu)$ where $t = (2\chi^2)^{\frac{1}{2}} - (2\nu)^{\frac{1}{2}}$, $x = (\nu/2)^{-\frac{1}{2}}$. $\Gamma(\chi^2, \nu)$, $\chi^2=0(.05)2(.1)10$, $\nu=0(.05)2(.1)6$; $Q(\chi^2/\nu)$, $\chi^2=0(.1)3.2$, $\nu=0(.05)2(.1)6$; $\chi^2=3.2(.2)7(.5)10(1)35$, $\nu=0(.1)4(.2)6$; $\mathcal{P}(t, \nu)$, $t=-4(.1)4.8$, $\nu=6(.5)11(1)32$; $\Pi(t, x)$: $t=-4.5(.1)4.8$, $x=0(.02).22(.01).25$, 5D.

Incomplete Beta Function, Binomial Distribution

- [26.33] Harvard University, Tables of the cumulative binomial probability distribution (Harvard Univ. Press, Cambridge, Mass., 1955).

$$\sum_{s=c}^n \binom{n}{s} p^s (1-p)^{n-s} \text{ for } p=.01(.01).5, 1/16, 1/12, 1/8, 1/6, 3/16, 5/16, 1/3, 3/8, 5/12, 7/16, n=1(1)50(2)100(10)200(20)500(50)1000, 5D.$$

- [26.34] National Bureau of Standards, Tables of the binomial probability distribution, Applied Math. Series 6 (U.S. Government Printing Office, Washington, D.C., 1950). $\binom{n}{s} p^s (1-p)^{n-s}$ and $\sum_{s=c}^n \binom{n}{s} p^s (1-p)^{n-s}$ for $p=.01(.01).5$, $n=2(1)49$, 7D.

- [26.35] K. Pearson (Editor), Tables of the incomplete beta function, Biometrika Office, University College (Cambridge Univ. Press, Cambridge, England, 1948). $I_x(a, b)$ for $x=.01(.01)1$; $a, b=.5(.5)11(1)50$, $a \geq b$, 7D.

- [26.36] W. H. Robertson, Tables of the binomial distribution function for small values of p , Office of Technical Services, U.S. Department of Commerce (1960).

$$\sum_{s=0}^c \binom{n}{s} p^s (1-p)^{n-s} \text{ for } p=.001(.001).02, n=2(1)100(2)200(10)500(20)1000; p=.021(.001).05, n=2(1)50(2)100(5)200(10)300(20)600(50)1000, 5D.$$

- [26.37] H. G. Romig, 50-100 Binomial tables (John Wiley & Sons, Inc., New York, N.Y., 1953).

$$\binom{n}{s} p^s (1-p)^{n-s} \text{ and } \sum_{s=0}^c \binom{n}{s} p^s (1-p)^{n-s} \text{ for } p=.01(.01).5 \text{ and } n=50(5)100, 6D.$$

- [26.38] C. M. Thompson, Tables of percentage points of the incomplete beta function, Biometrika 32, 151-181 (1941). Also reproduced as Table 16 in [26.11]. Tabulates values of x for which $I_x(a, b) = .005, .01, .025, .05, .1, .25, .5$; $2a=1(1)30, 40, 60, 120, \infty$; $2b=1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, 5D$.

- [26.39] U.S. Ordnance Corps, Tables of the cumulative binomial probabilities, ORDP 20-1, Office of Technical Services, Washington, D.C. (1952).

$$\sum_{s=c}^n \binom{n}{s} p^s (1-p)^{n-s} \text{ for } p=.01(.01).5 \text{ and } n=1(1)150, 7D.$$

F (Variance-Ratio) and Non-Central F Distribution

- [26.40] Table V of [26.7]. Tabulates values of F and

$$Z = \frac{1}{2} \ln F \text{ for } Q(F|v_1, v_2) = .2, .1, .05, .01, .001;$$

$$v_1=1(1)6, 8, 12, 24, \infty; v_2=1(1)30, 40, 60, 120, \infty, 2D \text{ for } F, 4D \text{ for } Z.$$

- [26.41] E. Lehmer, Inverse tables of probabilities of errors of the second kind, Ann. Math. Statist. 15, 388-398 (1944). $\phi = \sqrt{\lambda/(v_1+1)}$ for $v_1=1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$; $v_2=2(2)20, 24, 30, 40, 60, 80, 120, 240, \infty$ and $P(F'|v_1, v_2, \phi) = .2, .3$ where $Q(F'|v_1, v_2) = .01, .05, 3D \text{ or } 3S$.

- [26.42] M. Merrington and C. M. Thompson, Tables of percentage points of the inverted beta (F) distribution, Biometrika 33, 73-88 (1943). Tabulates values of F for which $Q(F|v_1, v_2) = .5, .25, .1, .05, .025, .01, .005$; $v_1=1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$; $v_2=1(1)30, 40, 60, 120, \infty$.

- [26.43] P. C. Tang, The power function of the analysis of variance tests with tables and illustrations of their use, Statistical Research Memoirs II, 126-149 and tables (1938). $P(F'|v_1, v_2, \phi)$ for $v_1=1(1)8, v_2=2(2)6(1)30, 60, \infty$ and $\phi = \sqrt{\lambda/v_1+1} = 1(.5)3(1)8$ where $Q(F'|v_1, v_2) = .01, .05, 3D$.

Student's t and Non-Central t -Distributions

- [26.44] E. T. Federighi, Extended tables of the percentage points of Student's t -distribution, J. Amer. Statist. Assoc. 54, 683-688 (1959). Values of

$$t \text{ for which } Q(t|\nu) = \frac{1}{2} [1 - A(t|\nu)] = .25 \times 10^{-n}, .1 \times 10^{-n}, n=0(1)6, .05 \times 10^{-n}, n=0(1)5, \nu=1(1)30(5)60(10)100, 200, 500, 1000, 2000, 10000, \infty; 3D.$$

- [26.45] Table III of [26.7]. Values of t for which $A(t|\nu) = .1(.1).9, .95, .98, .99, .999$ and $\nu=1(1)30, 40, 60, 120, \infty; 3D$.

- [26.46] N. L. Johnson and B. L. Welch, Applications of the noncentral t -distribution, Biometrika 31, 362-389 (1939). Tabulates an auxiliary function which enables calculation of δ for given t' and p , or t' for given δ and p where $P(t'|\nu, \delta) = p = .005, .01, .025, .05, .1(.1).9, .95, .975, .99, .995$.

- [26.47] J. Neyman and B. Tokarska, Errors of the second kind in testing Student's hypothesis, J. Amer. Statist. Assoc. 31, 318-326 (1936). Tabulates δ for $P(t'|\nu, \delta) = .01, .05, .1(.1).9$; $\nu=1(1)30, \infty$; $Q(t'|\nu) = .01, .05$.

- [26.48] Table 9 of [26.11]. $P(t|\nu) = \frac{1}{2} [1 + A(t|\nu)]$ for $t=0(.1)4(.2)8$; $\nu=1(1)20, 5D$; $t=0(.05)2(.1)4, 5$; $\nu=20(1)24, 30, 40, 60, 120, \infty, 5D$.

- [26.49] G. S. Resnikoff and G. J. Lieberman, Tables of the noncentral t -distribution (Stanford Univ. Press, Stanford, Calif., 1957). $\partial P(t'|\nu, \delta)/\partial t'$ and $P(t'|\nu, \delta)$ for $\nu=2(1)24(5)49, \delta = \sqrt{\nu+1} x_p$ where $Q(x_p) = p = .25, .15, .1, .065, .04, .025, .01, .004, .0025, .001$ and $t'/\sqrt{\nu}$ covers the range of values such that throughout most of the table the entries lie between 0 and 1, 4D.

Random Numbers and Normal Deviates

- [26.50] E. C. Fieller, T. Lewis and E. S. Pearson, Correlated random normal deviates, Tracts for Computers 26 (Cambridge Univ. Press, Cambridge, England, 1955).

- [26.51] T. E. Hull and A. R. Dobell, Random number generators, Soc. Ind. App. Math. 4, 230-254 (1962).

- [26.52] M. G. Kendall and B. Babington Smith, Random sampling numbers (Cambridge Univ. Press, Cambridge, England, 1939).

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| <p>[26.53] G. Marsaglia, Random variables and computers, Proc. Third Prague Conference in Probability Theory 1962. (Also as Math. Note No. 260, Boeing Scientific Research Laboratories, 1962).</p> <p>[26.54] M. E. Muller, An inverse method for the generation of random normal deviates on large scale computers, Math. Tables Aids Comp. 63. 167-174 (1958).</p> | <p>[26.55] Rand Corporation, A million random digits with 100,000 normal deviates (The Free Press, Glencoe, Ill. 1955).</p> <p>[26.56] H. Wold, Random normal deviates, Tracts for Computers 25 (Cambridge Univ. Press, Cambridge, England, 1948).</p> |
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